

4-5 LINEAR PROGRAMMING



As the T-shirt company grew, the students came to appreciate Hari more and more. He knows, both from experience and from studying, how mathematics helps in planning and evaluating business situations.

Hari was especially helpful when his friends learned that there are limits to what they can and cannot do in business. For example, they may have only between 10 and 15 hours a week available to work at their business, and during those hours they can do only so much work.

These limits, or constraints, are found in every business. The students buying coffee mugs or school supplies for resale discovered

that they could purchase only as many of these products as their money would allow. Sometimes they could not buy as many as they wanted because their suppliers had a limited number on hand to sell to them. They also discovered that there were limits to the demand for the products that they prepared. They might be able to sell 50 personalized lunch boxes, but they soon found that they could not expect to sell 500.

Smart businesspeople recognize the constraints under which their businesses must operate. They identify these constraints and make their plans within the boundaries of the constraints.

OBJECTIVES: *In this lesson, we will help Hari to:*

- Write inequalities for some of the business's constraints.
- Graph the inequalities and find the points of intersection.
- Test the points to find the numbers that minimize costs or maximize profits.

PROFIT

You have seen that, like all businesspeople, Evelyn and her friends want to make a profit. If they sell too few T-shirts, they will not cover their expenses. If they purchase too many, they will be left with unsold **inventory**, or stock on hand.

A more **efficient** business generally achieves greater profits at a lower cost. Thus consumers also benefit from efficiency.

CONSTRAINTS

Constraints are the conditions that limit business activities. Some examples of constraints are the amount of money available for investment, the time or materials available for production, and the demand for the product or service. There are many other constraints. Each business must discover its own limits and plan accordingly.

Linear programming is a mathematical method for planning within given constraints. The constraints are generally expressed as linear inequalities. A linear inequality is an inequality that has as its graph a half-plane bounded by a straight line.

Maximum and Minimum We use linear programming models to maximize or minimize certain factors in a business situation. To **maximize** means to find the greatest value within the constraints; to **minimize** means to find the least value within the constraints. For example, a business wishes to maximize profits and minimize costs.

Ask Yourself

1. What are three of the constraints that a person starting a business must consider?
2. What does a business try to minimize?
3. What does a business try to maximize?
4. How does a business become more efficient?

ALGEBRA REVIEW

Graph each equation.

1. $y = 2x - 3$
2. $3x + 2y = 24$

Find the coordinates of the point of intersection.

3. $x + y = 5$
 $2x - y = 4$
4. $3y = 5x - 1$
 $y - x = 1$

Solve by graphing.

5. $x + 2y \geq 2$
 $2x - y \leq 3$
6. $x \leq 3y$
 $2y - 1 \geq x$
7. Solve the system by graphing.
Find the coordinates of the region contained by the inequalities.
 $x \leq 4$
 $y \leq 6$
 $x + 2y \geq 12$

SHARPEN YOUR SKILLS

SKILL 1

EXAMPLE 1 Evelyn and her family are planning a car trip that will take several days. They must consider the following constraints.

- On the first day they are leaving at about noon and don't want to drive after dark. They have only 7 hours in which to drive.
- They want to travel at least 300 miles that day.
- They want to stay within the speed limit of 55 miles per hour.



QUESTION How can you use linear programming to graph the constraints as inequalities and show different possible ways to make the trip?

SOLUTION

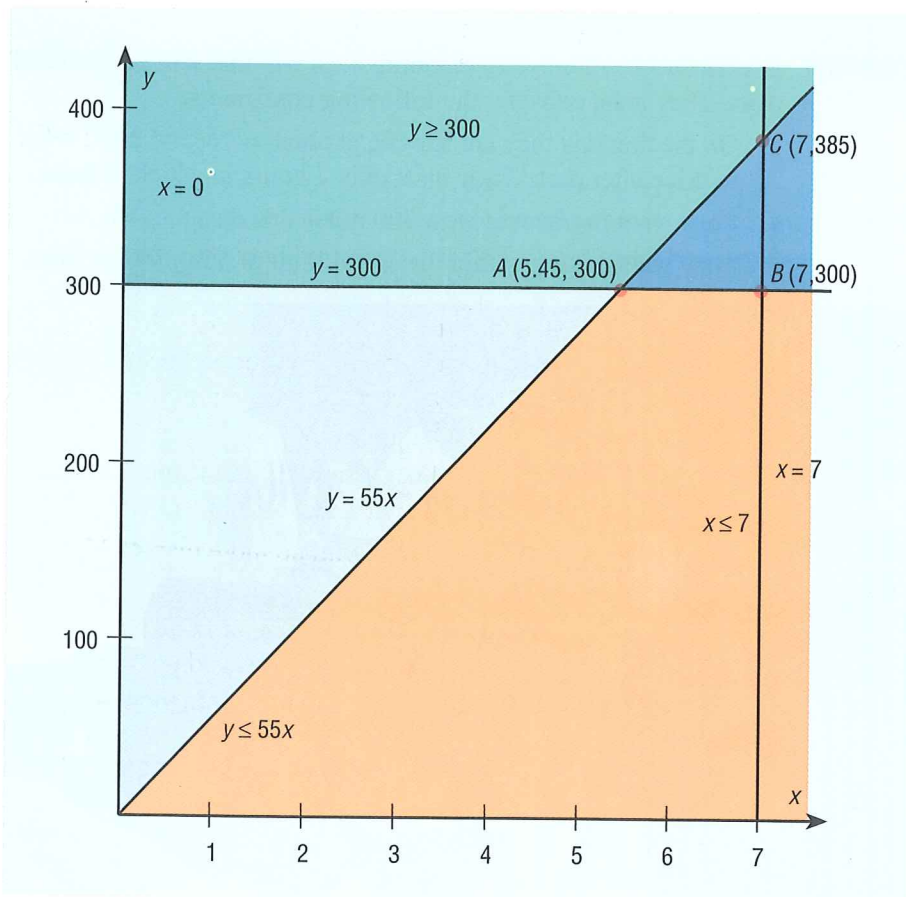
Let x represent hours and y represent miles. Write inequalities for each constraint.

- The trip must be no more than 7 hours. $x \leq 7; x \geq 0$
- The distance must be at least 300 miles. $y \geq 300$
- Obey the speed limit of 55 miles per hour. $\text{Distance} = \text{rate} \cdot \text{time}$
 $y \leq 55x$

Note that x must be greater than zero, since time is positive.



Graph the system of inequalities as shown. The triangular shaded region satisfies all three inequalities. You may not be able to graph $x \leq 7$ on your graphing calculator since it is not a function.



We can find the corners of the triangular region by solving each pair of equations that intersect at the given point.

Point A

$$y = 300$$

$$y = 55x$$

$$55x = 300$$

$$x = 5.45$$

$$y = 300$$

$$A(5.45, 300)$$

Point B

$$y = 300$$

$$x = 7$$

$$B(7, 300)$$

Point C

$$x = 7$$

$$y = 55x$$

$$y = 55x$$

$$y = 55(7)$$

$$y = 385$$

$$C(7, 385)$$

Although all the points in the shaded region satisfy the three inequalities, the vertices have special meaning.

$A(5.45, 300)$ represents the shortest driving time, 5 h 27 min. It would give the least amount of time in the car.

$B(7, 300)$ represents the lowest speed possible, 43 miles per hour. It would offer the most leisurely drive.

$C(7, 385)$ represents the greatest distance, 385 miles. You would use this to travel as far as possible before dark.

SKILL 2

EXAMPLE 2 Evelyn, Freda, Greg, and Hari have begun selling both T-shirts and sweatshirts. They purchase the T-shirts for \$5.50 and the sweatshirts for \$7.50. They have become successful but have also discovered constraints that affect their business.

- To satisfy the demand and not disappoint customers, they must produce a total of at least 52 shirts a week.
- Their supplier can supply them with no more than 45 blank sweatshirts per week.
- Because of the coming cool weather, they must be prepared to sell at least as many sweatshirts as T-shirts, and possibly more.
- Because of the time available, they cannot prepare more than a total of 70 shirts of both kinds per week.

QUESTION How can they use linear programming to find the lowest costs within the given constraints?

SOLUTION

To solve the problem using linear programming, you must first do two things:

- Choose a quantity that you want to maximize or minimize and write an equation for that quantity. This quantity is called the **objective function**.
- Identify the constraints.

In this case we want to minimize the total cost of the T-shirts and sweatshirts. This cost is expressed by the following equation.

$$c = 5.50x + 7.50y \quad \text{where } c = \text{the cost}$$
$$x = \text{the number of T-shirts}$$
$$y = \text{the number of sweatshirts}$$

Expressing the constraints as inequalities, we have the following.

- a. The students must sell a total of more than 52 shirts.

$$x + y \geq 52$$

In slope-intercept form, this is written as

$$y \geq -x + 52$$

- b. They can obtain no more than 45 blank sweatshirts.

$$y \leq 45$$

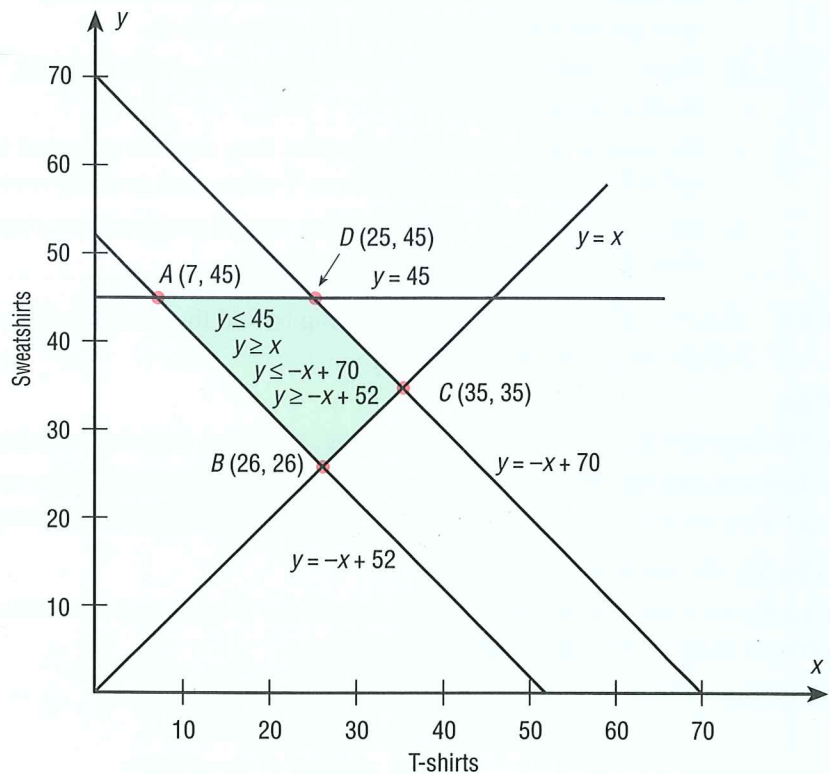
- c. They will sell at least as many sweatshirts as T-shirts.

$$y \geq x$$

- d. They cannot prepare more than a total of 70 shirts.

$$y + x \leq 70 \quad \text{or} \quad y \leq -x + 70$$

We now graph the lines for each of these inequalities. This can be done by using a graphing calculator.



The shaded region satisfies all four inequalities.

Solve each pair of equations that intersect at the given point.

<i>Point A</i>	<i>Point B</i>	<i>Point C</i>	<i>Point D</i>
$y = 45$	$y = x$	$y = x$	$y = 45$
$x + y = 52$	$x + y = 52$	$x + y = 70$	$x + y = 70$
$x + 45 = 52$	$2x = 52$	$2x = 70$	$x + 45 = 70$
$x = 7, y = 45$	$x = 26, y = 26$	$x = 35, y = 35$	$x = 25, y = 45$
$A(7, 45)$	$B(26, 26)$	$C(35, 35)$	$D(25, 45)$

Substitute these coordinates in the objective function, $c = 5.50x + 7.50y$.

For point <i>A</i> ,	$5.50(7) + 7.50(45) = 376$
For point <i>B</i> ,	$5.50(26) + 7.50(26) = 338$
For point <i>C</i> ,	$5.50(35) + 7.50(35) = 455$
For point <i>D</i> ,	$5.50(25) + 7.50(45) = 475$

The lowest cost will be achieved by using the coordinates at point *B*; that is, by making and selling 26 T-shirts and 26 sweatshirts.

SKILL 3

The constraints listed and graphed in Example 2 will be the same whether we are considering cost or revenue.

EXAMPLE 3 Evelyn's group can raise some additional money to invest, so they want to maximize profit. They sell the T-shirts for \$17 each and the sweatshirts for \$23 each.

QUESTION How can they find the sales quantities that will give the maximum profit within the given constraints?

SOLUTION

We use the same linear programming model that was used in Example 2. However, we use a different objective function for revenue. This is the revenue function, the number of items sold multiplied by unit price.

$$r = 17x + 23y \quad \text{where } 17 = \text{the selling price of the T-shirts}$$

$$x = \text{the number of T-shirts sold}$$

$$23 = \text{the selling price of the sweatshirts}$$

$$y = \text{the number of sweatshirts sold}$$

For intersection points on the graph the revenues are

$$A(7, 45): \quad 17(7) + 23(45) = 1154$$

$$B(26, 26): \quad 17(26) + 23(26) = 1040$$

$$C(35, 35): \quad 17(35) + 23(35) = 1400$$

$$D(25, 45): \quad 17(25) + 23(45) = 1460$$

The maximum revenue is at point *D*. But this is not the maximum profit.

Profit equals revenue minus cost: $p = r - c$.

We now use the costs from Example 2 to find profit at each point.

	<i>revenue</i>	—	<i>cost</i>	=	<i>profit</i>
$A(7, 45)$	1154	—	376	=	778
$B(26, 26)$	1040	—	338	=	702
$C(35, 35)$	1400	—	455	=	945
$D(25, 45)$	1460	—	475	=	985

Within the given constraints the greatest profit occurs at point $D(25, 45)$, which represents 25 T-shirts and 45 sweatshirts. We can also find the maximum profit by writing an objective function for profit.

$$p = 17x - 23y - (5.50x + 7.50y)$$

$$p = 11.5x + 15.5y$$



TRY YOUR SKILLS

1. Find the coordinates of the point of intersection of the graphs $y = 8$ and $3x + y = 20$.
2. Draw the triangle bounded by the inequalities $y \leq 8$, $x \leq 12$, $2y + x \geq 20$.
3. Find the vertices of the triangle drawn in Exercise 2.
4. Write inequalities for the following constraints.
 - a. You want to work no more than 10 hours per week.
 - b. You want to earn at least \$45 per week.
 - c. You won't be able to get a job earning more than \$6.00 per hour.
5. Draw the triangle for the constraints listed in Exercise 4, and explain what each vertex of the triangle means.



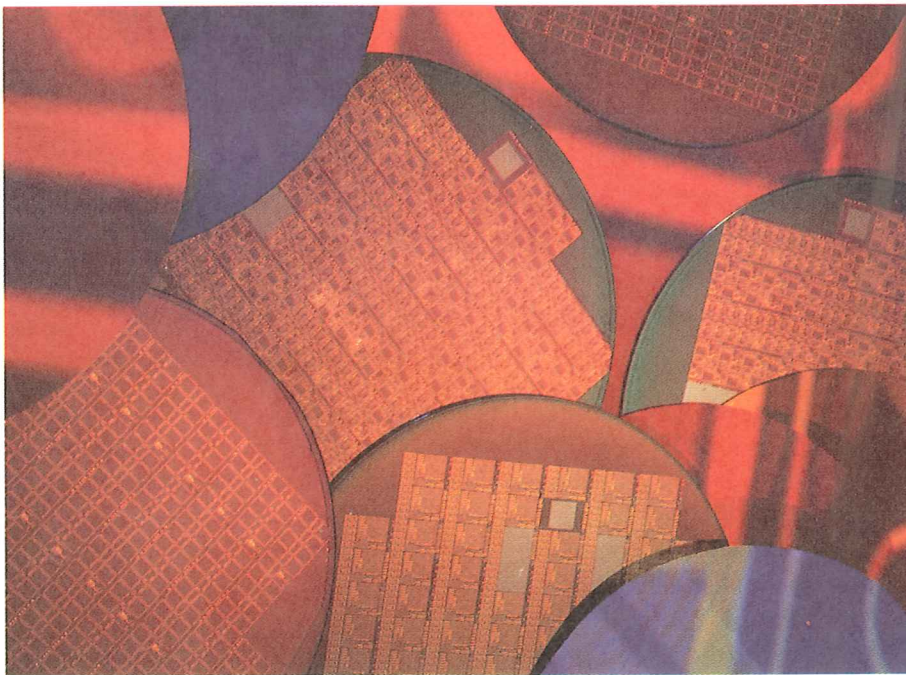
EXERCISE YOUR SKILLS

Students selling pennants have joined those selling bumper stickers. They can purchase pennants for \$1.00 each and bumper stickers for \$1.25 each. They want to earn as much money as possible but have discovered that they must work under the following constraints. Let x represent pennants, and y represent bumper stickers.

- a. They can obtain no more than 200 pennants.
 - b. They can obtain no more than 400 bumper stickers.
 - c. They must sell a combined total of at least 300.
1. Write inequalities to express the constraints. Let x represent the number of pennants and y represent the number of bumper stickers.

2. Graph the constraint inequalities, and shade the region that satisfies all the inequalities.
3. Find the coordinates of the corners of the region in Exercise 2.
4. Write the objective function to minimize costs.
5. Write the objective function to maximize revenue if the pennants and bumper stickers sell for \$2 each.
6. Write the objective function to maximize profit.
7. Substitute the corner coordinates in the objective function to find the number of pennants and stickers that they should sell to maximize profit.

A friend of Hari's goes to work for a computer company that manufactures computer chips. To meet demands, the company must manufacture at least 4000 chips of type A per day and 6000 chips of type B per day. The testing department cannot process more than a total of 15,000 per day. The company earns a profit of \$0.50 for each chip of type A sold and \$0.40 for each chip of type B sold. Let x stand for the number of A chips and y stand for the number of B chips.



8. Write the objective function for the profit.
9. Write three inequalities to express the constraints.
10. Graph the inequalities.
11. Find the coordinates of the corners of the region defined by the inequalities.
12. Test the coordinates of the corners in the objective function to find the pair that gives the greatest profit.

KEY TERMS

constraints
efficient
inventory
linear programming
maximize
minimize
objective function

Students selling coffee mugs have joined with those selling desk organizers. Costs are \$3.55 for each coffee mug and \$6.40 for each desk organizer. To make a profit, they must sell a combined total of at least 50 coffee mugs and desk organizers. The total spent on these supplies for both items must be less than \$350. Because of arrangements with suppliers, the number of mugs must be less than four times the number of organizers. The number of organizers must be less than 40. Let x represent the coffee mugs and let y represent the organizers.

13. Write the objective function to minimize costs.
14. Write inequalities to express the constraints.
15. Graph the constraint inequalities and shade the region that satisfies all the inequalities.
16. Find the corners of the region in Exercise 15.
17. What is the number of coffee mugs and desk organizers they should sell to minimize their cost?

Freda's father is a farmer with 210 acres on which to plant. From crop x he will earn \$400 an acre; from crop y he will earn \$350 an acre. To meet demands, he must plant at least 40 acres of crop x and 50 acres of crop y . Because of the conditions of the soil, he may not plant more than 80 acres of crop x . According to state regulations, the acres for crop y must be no more than twice the number of acres for crop x .

18. Write the objective function to maximize revenue.
19. Write inequalities to express the constraints.
20. Graph the constraint inequalities and shade the region that satisfies all the inequalities.
21. Find the corners of the region in Exercise 20.
22. How many acres of each crop should he plant to maximize his earnings?

Some students who are starting a business plan to sell earrings for \$4 a pair and necklaces for \$10 each. Their unit costs are \$2.50 for the earrings and \$9 for the necklaces. They must work within the following constraints. The store through which they are selling will take only up to a total of 45 pairs of earrings and necklaces combined. Their source of supplies is such that the number of necklaces that they can make will be less than twice the number of pairs of earrings. They must sell at least 10 necklaces to maintain their agreement with the store. Demand will not permit them to sell more than 30 pairs of earrings. Let x represent the earrings and y represent the necklaces.

23. Write the objective function to minimize cost.
24. Write the objective function to maximize revenue.
25. Write the objective function to maximize profit.
26. Write inequalities to express the constraints.
27. Graph the constraint inequalities and shade the region that satisfies all the inequalities.
28. Find the corners of the region in Exercise 27.

29. How many of each should they sell to minimize cost?
30. How many of each should they sell to maximize revenue?
31. How many of each should they sell in order to maximize profit?

MIXED REVIEW

Joseph repairs watches at \$10 an hour as well as \$2 for each watch that he repairs. He works 30 hours a week.

1. How many watches does he have to repair in order to earn at least \$350 per week?
2. It takes Joseph an average of 20 minutes to repair a watch. What is the most money that he can reasonably expect to make in one week?

Suppose that \$100 is invested at 5% annually. Find the value of the investment after the given number of years and at the given frequency of compounding. Use the calculator's exponent key $\boxed{x^y}$.

3. 50 years, compounded annually
4. 50 years, compounded quarterly
5. 100 years, compounded annually

A partially completed monthly payroll summary is shown below. Complete the table. Use a spreadsheet program, if available.

	Week Ending	Gross Pay	Income Tax Withholding	FICA Withholding	Take-Home Pay
6.	9/3	\$ 87.00	\$10		
7.	9/10	123.00	19		
8.	9/17	167.50	23		
9.	9/24	201.00	28		
10.	Totals				

11. What is the limit of your liability if you immediately report the loss or theft of your ATM card?
12. The fixed costs in Amanda's business are \$325 per week. The variable costs are \$3.50 per item. What is her cost if she produces 150 items in one week?

CHAPTER 4 REVIEW

Find the following earnings:

	Number of Hours	Hourly Rate	Weekly Earnings		Number of Hours	Hourly Rate	Weekly Earnings
1.	38	\$ 3.35		2.	26	\$ 7.50	
3.	41	22.70		4.	22	14.80	

Find the weekly take-home pay for each of these single people if each claims one exemption. Use the tables for federal income tax withholding in the Reference Section.

	Gross Pay	Income Tax Withholding	FICA Withholding	Take-Home Pay
5.	\$127.30			
6.	195.00			
7.	930.70			
8.	325.60			

Duanita and Alisha want to make and sell stuffed kittens. Use the information given in the box to help you find the total production costs for manufacturing the amounts of toys in Exercises 9, 10, and 11.

Fixed Costs		Variable Costs	
Labor	\$3.50 per hour for 30 hours	Materials	\$2.25 each
Advertising	\$3.00	Packaging	\$0.02 each
Energy	\$3.58		
Transportation	\$5.92		

9. 20 kittens 10. 40 kittens 11. 60 kittens

Duanita and Alisha will sell stuffed kittens for \$7.00 each. Use the costs in Exercises 9-11 to find the profit or loss from selling each amount.

12. 20 kittens 13. 40 kittens 14. 60 kittens

Use your results from Exercises 9-14 to make a spreadsheet that shows the following information for producing and selling 20, 40, and 60 kittens.

	Fixed Cost	Unit Cost	Number Produced	Total Cost	Unit Price	Revenue	Profit (Loss)
15.			20				
16.			40				
17.			60				

18. Write the cost function.
19. Write the revenue function.
20. Write the profit function.
21. Graph the cost function.
22. Graph the revenue function on the same set of axes as the cost function.
23. Find the break-even point algebraically and using graphing techniques.
24. Why is it important to know how much profit (or loss) a business makes?
25. If a company does not make enough money to break even, what kinds of changes should be considered? Write a paragraph explaining the changes you would consider if your company were losing money and what you would do next if each strategy did not work.
26. After one year, the employees in Oscar's company expected to receive a raise in salary. Explain how an employee's thinking about a raise in salary might differ from the owner's thinking. What should Oscar consider before deciding whether to give raises? Why might it be better for the company and for the employees themselves if salary raises are small? When might salary raises be large?

Use the following information to do Exercises 27–30. When Duanita and Alisha were successful in the stuffed kitten business, they expanded and began to make stuffed puppies as well. Their cost for materials for the kittens continued to be \$2.27, but the materials for the puppies cost \$3.00. They sold the kittens for \$7.00 and the puppies for \$7.50. To meet the demand, they had to sell more kittens than puppies. They could not obtain materials to make more than 40 kittens per week and more than 20 puppies per week. To pay their expenses, they had to sell a total of more than 50 per week. Let x represent the stuffed kittens and y represent the stuffed puppies.

27. Write the objective function to minimize cost.
28. Write the objective function to maximize revenue.
29. Write the objective function to maximize profit.
30. Write inequalities for the constraints.
31. Graph the constraints and shade the region that satisfies all the inequalities.
32. Find the corners of the region in Exercise 31.
33. Find the quantities that will minimize cost within the constraints.
34. Find the quantities that will maximize revenue within the constraints.
35. Find the quantities that will maximize profit within the constraints.

CHAPTER 4 TEST

Find the weekly earnings for each.

1. 42 hours at \$4.80 per hour 2. 33 hours at \$17.25 per hour

Find the weekly take-home pay for each gross pay. Each person is single and claims one withholding allowance (exemption). Tables for federal income tax withholding are in the Reference Section.

3. \$201.60 4. \$569.25 5. \$202.50 6. \$844.20

Akira sells stadium cushions. Find the total production cost to manufacture the numbers of cushions indicated in Exercises 7–9.

Fixed Costs		Variable Costs	
Labor	\$4.25 per hour for 30 hours	Materials	\$1.40 each
		Packaging	\$0.50 each
Advertising	\$2.00		
Energy	\$2.58		
Transportation	\$4.42		

7. 30 cushions 8. 60 cushions 9. 90 cushions

Find the profit or loss for selling the following amounts at \$5.50 each.

10. 30 cushions 11. 60 cushions 12. 90 cushions

Use your results from the previous exercises to complete the spreadsheet.

	Fixed Cost	Unit Cost	Number Produced	Total Cost	Unit Price	Revenue	Profit (Loss)
13.			30				
14.			60				
15.			90				

16. Graph the cost and revenue functions.
17. Find the break-even point algebraically.

Jod and Jessie sell stadium cushions and caps. The cushions cost \$1.90; the caps cost \$2.25. They sell the cushions for \$5.00 and the caps for \$6.00. They can obtain no more than 100 cushions and 75 caps per week. To meet demands, they have to sell a total of at least 120 of the two together. They cannot package more than 150 per week. Let x represent the cushions and y represent the caps.

18. Use linear programming to graph the given constraints, and find the points that will give maximum and minimum quantities.
19. What quantities will give the minimum cost?
20. What quantities will give the maximum revenue?
21. What quantities will give the maximum profit?

CUMULATIVE REVIEW

1. Your regular rate is \$6.22 per hour. You also receive $1\frac{1}{2}$ times your hourly rate for any hours over 20 in one week. Last week you worked 26 hours. You also received \$12.50 in tips. Find your total wages for the week.
2. You work according to the time schedule shown. You earn \$5 per hour. Find your total earnings for the week if you receive \$22 in tips that week.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
3–6	3–7	3–6	3–6	4–9	8–12

3. You earn \$9.30 per hour. You work 35 hours a week for 50 weeks and get 2 weeks of paid vacation as well as 18% of your base salary in fringe benefits. Find your yearly earnings including benefits.
4. You are a real-estate agent and sold a house for \$110,000. You earn a 4% commission on the first \$100,000 of each sale and 6% commission on the amount over \$100,000. How much commission did you earn on the sale?
5. You earned \$94.00. Your income tax withholding is \$13 and your FICA withholding is 7.65% of your earnings. Find your take-home pay.
6. You wrote 33 checks one month. The bank charges \$0.0225 for the first 20 checks that you write and \$0.08 for each written check over 20. How much are the charges for that month?
7. Your employer made out your paycheck with your first and middle initials and full last name. Write your name as you would to endorse the check.
8. You save \$28 per week. You want to buy a stereo system that costs \$589. For how many weeks must you save before you can have the stereo system?
9. Felix withdraws the interest from his savings account as soon as the interest is posted twice a year. What is the simple interest earned in $2\frac{1}{2}$ years on his \$500 balance? The annual interest rate is 3.5%.
10. Find the value of an investment that is originally worth \$1,000 after it has been compounded quarterly for 1 year at an 8.5% annual rate.
11. Suppose that the reserve requirement on banking deposits is 12%. What will be the multiplier that corresponds to this requirement?
12. The fixed cost of producing hand-painted sweatshirts is \$285 per week. The variable costs are \$8 per shirt and \$1 per shirt for paints. What is the total cost of producing 25 shirts in one week?
13. Use the information in Exercise 12 to find the profit or loss, given that each sweatshirt is sold for \$20.
14. Write the equations for the cost function and revenue function of Exercises 12 and 13. Solve these equations to find the break-even point for the sweatshirt-selling operation.



PROJECT 4-1: Retail Sales

Work in small groups to set up imaginary companies. Select items that your company would like to sell or services that it would provide. You may use information from the chapter to plan costs or do your own research to find the costs of producing various services or products. Tell how you arrived at these costs for each product or service that your company produces.

1. Decide on the product or service for your company. Describe that product or service. Be sure to include all elements including advertising, energy use, transportation, and packaging.
2. Decide on the number of hours per week each employee (member of the group) will work. Decide the hourly wages for each person. Decide the marital status and number of exemptions for each employee.
3. Complete a payroll register showing each employee's earnings for four consecutive weeks.
4. Furnish a four-week payroll summary for each employee.
5. List your fixed and variable costs. Use a spreadsheet to show fixed costs, variable costs, number produced, total costs, average sale (or sale per unit), total sales, and profit or loss. Your spreadsheet should have entries that result in a loss and others that result in a profit.
6. Write the cost, revenue, and profit functions for your product.
7. Construct a line graph showing the break-even point for the business.
8. List several realistic constraints related to quantities and costs. Use linear programming to minimize costs or maximize sales revenue within the given constraints.

Extensions

1. After you have completed a plan for your company, get together with other groups in your class and try to sell them your product.
2. On the basis of feedback from other groups, redesign your product or service to be more useful or attractive to potential customers.
3. Listen in turn to other groups' presentations, and give them feedback about their product or service. Try to make comments that will help them in redesigning their product or rethinking their service.