UNIT 5

Calculus

Calculus is one of the most important areas of mathematics. There are two branches of calculus, *differential* calculus and *integral* calculus. Differential calculus deals mainly with variable, or changing, quantities. Integral calculus deals mainly with finding sums of infinitesimally small quantities. This generally involves finding a limit.

Chapter 15, the only chapter in Unit 5, provides an overview of some aspects and applications of calculus.

Chapter 15 Introduction to Calculus

CHAPTER OBJECTIVES

- **Evaluate limits of functions.** *(Lesson 15-1)*
- **Find derivatives and antiderivatives of polynomial functions.** *(Lessons 15-2, 15-4)*
- **Evaluate definite integrals using limits and the Fundamental Theorem of Calculus.** *(Lessons 15-3, 15-4)*

WORLD WIDE WEB

Unit 5 interNET Project

DISEASES

Did you know that many communicable diseases have been virtually eliminated as the result of vaccinations? In 1954, Jonas Salk invented a vaccine for polio. Polio was a dreaded disease from about 1942 to 1954. In 1952, there were 60,000 cases reported. As a result of Salk's miraculous discovery, there were only 5 cases of polio reported in the United States in 1996. In this project, you will look at data about dise… the United States.

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Miracles of Science! Even though many diseases that once disabled or even killed many people have been controlled, the treatment or cure for many other diseases still eludes researches. Use the Internet to find data on a particular disease.
Math Connection: Model the data with at least two functions. Predict the course of the disease in the future using your model.

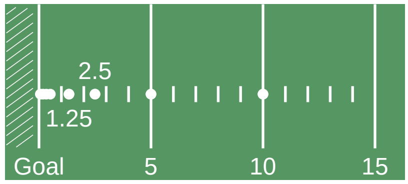
interNET CONNECTION For more information on the Unit Project, visit: www.amc.glencoe.com

15-1 Limits

OBJECTIVES
- Calculate limits of polynomial and rational functions algebraically.
- Evaluate limits of functions using a calculator.

Real World Application

SPORTS In football, if the length of a penalty exceeds half the distance to the offending team's goal line, then the ball is moved only half the distance to the goal line. Suppose one team has the ball at the other team's 10-yard line. The other team, in an effort to prevent a touchdown, repeatedly commits penalties. After the first penalty, the ball would be moved to the 5-yard line.



The results of the subsequent penalties are shown in the table. Assuming the penalties could continue indefinitely, would the ball ever actually cross the goal line?

Penalty	1st	2nd	3rd	…
Yard Line	5	2.5	1.25	…

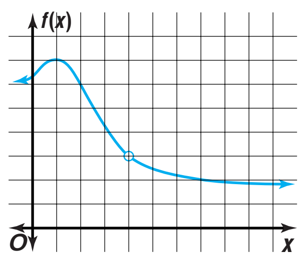
The ball will never reach the goal line, but it will get closer and closer after each penalty. As you saw in Chapter 12, a number that the terms of a sequence approach, without necessarily reaching it, is called a **limit**. In the application above, the limit is the goal line or 0-yard line. The idea of a limit also exists for functions.

Limit of a Function

If there is a number L such that the value of $f(x)$ gets closer and closer to L as x gets closer to a number a, then L is called the limit of $f(x)$ as x approaches a.

In symbols, $L = \lim_{x \to a} f(x)$.

Example 1 **Consider the graph of the function $y = f(x)$ shown at the right. Find each pair of values.**



a. $f(2)$ and $\lim_{x \to 2} f(x)$

At the point on the graph where the x-coordinate is 2, the y-coordinate is 6. So, $f(2) = 6$.

Look at points on the graph whose x-coordinates are close to, but not equal to, 2. Notice that the closer x is to 2, the closer y is to 6. So, $\lim_{x \to 2} f(x) = 6$.

b. $f(4)$ and $\lim_{x \to 4} f(x)$

The hole in the graph indicates that the function does not have a value when $x = 4$. That is, $f(4)$ is undefined.

Look at points on the graph whose x-coordinates are close to, but not equal to, 4. The closer x is to 4, the closer y is to 3. So, $\lim_{x \to 4} f(x) = 3$.

You can see from Example 1 that sometimes $f(a)$ and $\lim_{x \to a} f(x)$ are the same, but at other times they are different. In Lesson 3-5, you learned about continuous functions and how to determine whether a function is continuous or discontinuous for a given value. We can use the definition of continuity to make a statement about limits.

Limit of a Continuous Function

$f(x)$ is continuous at a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Examples of continuous functions include polynomials as well as the functions $\sin x$, $\cos x$, and a^x. Also, $\log_a x$ is continuous if $x > 0$.

Example 2 **Evaluate each limit.**

a. $\lim_{x \to 3} (x^3 - 5x^2 + 7x - 10)$

Since $f(x) = x^3 - 5x^2 + 7x - 10$ is a polynomial function, it is continuous at every number. So the limit as x approaches 3 is the same as the value of $f(x)$ at $x = 3$.

$$\begin{aligned}\lim_{x \to 3} (x^3 - 5x^2 + 7x - 10) &= 3^3 - 5 \cdot 3^2 + 7 \cdot 3 - 10 && \textit{Replace x with 3.}\\ &= 27 - 45 + 21 - 10\\ &= -7\end{aligned}$$

The limit of $x^3 - 5x^2 + 7x - 10$ as x approaches 3 is -7.

b. $\lim_{x \to \pi} \frac{\cos x}{x}$

Since the denominator of $\frac{\cos x}{x}$ is not 0 at $x = \pi$, the function is continuous at $x = \pi$.

$$\begin{aligned}\lim_{x \to \pi} \frac{\cos x}{x} &= \frac{\cos \pi}{\pi} && \textit{Replace x with } \pi.\\ &= \frac{-1}{\pi} && \cos \pi = -1\end{aligned}$$

The limit of $\frac{\cos x}{x}$ as x approaches π is $-\frac{1}{\pi}$.

Limits can also be used to model real-world situations in which values approach a given value.

Example 3 Real World Application

PHYSICS **According to the special theory of relativity developed by Albert Einstein, the length of a moving object, as measured by an observer at rest, shrinks as its speed increases. (The difference is only noticeable if the object is moving very fast.) If L_0 is the length of the object when it is at rest, then its length L, as measured by an observer at rest, when traveling at speed v is given by the formula $L = L_0\sqrt{1 - \frac{v^2}{c^2}}$, where c is the speed of light. If the space shuttle were able to approach the speed of light, what would happen to its length?**

interNET CONNECTION

Research
For more information about relativity, visit: **www.amc.glencoe.com**

We need to find $\lim_{v \to c} L_0\sqrt{1 - \frac{v^2}{c^2}}$.

$$\lim_{v \to c} L_0\sqrt{1 - \frac{v^2}{c^2}} = L_0\sqrt{1 - \frac{c^2}{c^2}} \quad \textit{Replace } v \textit{ with } c\textit{, the speed of light.}$$

$$= L_0\sqrt{0}$$

$$= 0$$

The closer the speed of the shuttle is to the speed of light, the closer the length of the shuttle, as seen by an observer at rest, gets to 0.

When a function is not continuous at the x-value in question, it is more difficult to evaluate the limit. Consider the function $f(x) = \frac{x^2 - 9}{x - 3}$. This function is not continuous at $x = 3$, because the denominator is 0 when $x = 3$. To compute $\lim_{x \to 3} f(x)$, apply algebraic methods to decompose the function into a simpler one.

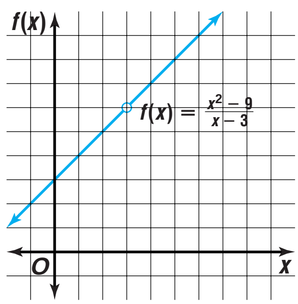
$$\frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{x - 3} \quad \textit{Factor.}$$

$$= x + 3, x \neq 3 \quad \textit{Simplify.}$$

When computing the limit, we are only interested in x-values close to 3. What happens when $x = 3$ is irrelevant, so we can replace $f(x)$ with the simpler expression $x + 3$.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} (x + 3)$$

$$= 3 + 3 \text{ or } 6$$



The graph of $f(x)$ indicates that this answer is correct. As x gets closer to 3, the y-coordinates get closer and closer to, but never equal, 6. The limit is 6.

Example 4 **Evaluate each limit.**

a. $\lim\limits_{x \to 4} \dfrac{x^2 - 2x - 8}{x^2 - 4x}$

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x^2 - 4x} = \lim_{x \to 4} \frac{(x+2)(x-4)}{x(x-4)}$$

$$= \lim_{x \to 4} \frac{x+2}{x}$$

$$= \frac{4+2}{4} \text{ or } \frac{3}{2} \quad \textit{Replace } x \textit{ with } 4.$$

b. $\lim\limits_{h \to 0} \dfrac{h^3 - 4h^2 - 6h}{h}$

$$\lim_{h \to 0} \frac{h^3 - 4h^2 - 6h}{h} = \lim_{h \to 0} \frac{h(h^2 - 4h - 6)}{h}$$

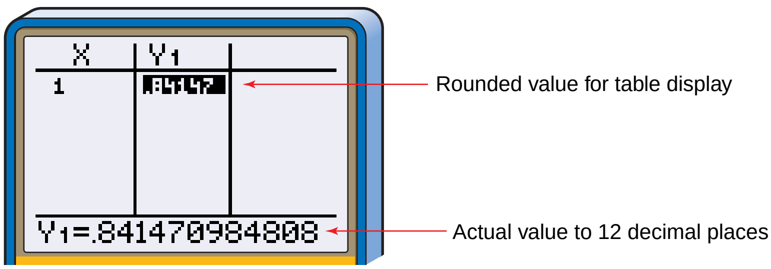
$$= \lim_{h \to 0} (h^2 - 4h - 6)$$

$$= 0^2 - 4 \cdot 0 - 6 \text{ or } -6 \quad \textit{Replace } h \textit{ with } 0.$$

Sometimes algebra is not sufficient to find a limit. A calculator may be useful. Consider the problem of finding $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$, where x is in radians. The function is not continuous at $x = 0$, so the limit cannot be found by replacing x with 0. On the other hand, the function cannot be simplified to help make the limit easier to find. You can use a calculator to compute values of the function $\dfrac{\sin x}{x}$ for x-values that get closer and closer to 0 from either side (that is, both less than 0 and greater than 0).

Graphing Calculator Tip

Enter the function in the **Y=** menu and set **Indpnt** to **Ask** in the **TBLSET** menu to help generate these values.



The tables below show the expression evaluated for values of x that approach 0.

x	$\dfrac{\sin x}{x}$
1	0.841470984808
0.1	0.998334166468
0.01	0.999983333417
0.001	0.999999833333
0.0001	0.999999998333

x	$\dfrac{\sin x}{x}$
−1	0.841470984808
−0.1	0.998334166468
−0.01	0.999983333417
−0.001	0.999999833333
−0.0001	0.999999998333

As x gets closer and closer to 0, from either side, the value of $\dfrac{\sin x}{x}$ gets closer and closer to 1. That is, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

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Example 5 **Evaluate each limit.**

a. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ (*x* is in radians.)

x	$\frac{1 - \cos x}{x^2}$
1	0.45970
0.1	0.49958
0.01	0.499996
0.001	0.49999996

x	$\frac{1 - \cos x}{x^2}$
−1	0.45970
−0.1	0.49958
−0.01	0.499996
−0.001	0.49999996

A graphing calculator or spreadsheet can generate more decimal places for the expression than shown here.

As x approaches 0, the value of $\frac{1 - \cos x}{x^2}$ gets closer to 0.5, so $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = 0.5$.

b. $\lim_{x \to 1} \frac{\ln x}{x - 1}$

x	$\frac{\ln x}{x - 1}$
0.9	1.0536
0.99	1.0050
0.999	1.0005

x	$\frac{\ln x}{x - 1}$
1.1	0.95310
1.01	0.99503
1.001	0.99950

The closer x is to 1, the closer $\frac{\ln x}{x - 1}$ is to 1, so $\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$.

Using a calculator is not a foolproof way of evaluating $\lim_{x \to a} f(x)$. You may only analyze the values of $f(x)$ for a few values of x near a. However, the function may do something unexpected as x gets even closer to a. You should use algebraic methods whenever possible to find limits.

GRAPHING CALCULATOR EXPLORATION

You can use a graphing calculator to find a limit, with less work than an ordinary scientific calculator. To find $\lim_{x \to a} f(x)$, first graph the equation $y = f(x)$. Then use ZOOM and TRACE to locate a point on the graph whose x-coordinate is as close to a as you like. The y-coordinate should be close to the value of the limit.

TRY THESE **Evaluate each limit.**

1. $\lim_{x \to 0} \frac{e^x - 1}{x}$

2. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 3x + 2}$

WHAT DO YOU THINK?

3. If you graph $y = \frac{\ln x}{x - 1}$ and use TRACE, why doesn't the calculator tell you what y is when $x = 1$?

4. Solve Exercise 2 algebraically. Do you get the same answer as you got from the graphing calculator?

5. Will the graphing calculator give you the exact answer for every limit problem? Explain.

CHECK FOR UNDERSTANDING

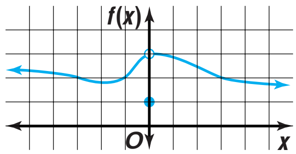
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Define** the expression *limit of f(x) as x approaches a* in your own words.
2. **Describe** the difference between $f(1)$ and $\lim_{x \to 1} f(x)$ and explain when they would be the same number.
3. *Math Journal* **Write** a description of the three methods in this lesson for computing $\lim_{x \to a} f(x)$. Explain when each method would be used and include examples.

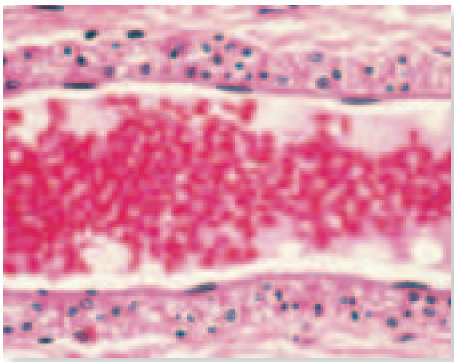
Guided Practice

4. Use the graph of $y = f(x)$ to find $\lim_{x \to 0} f(x)$ and $f(0)$.



Evaluate each limit.

5. $\lim_{x \to 2} (-4x^2 + 2x - 5)$
6. $\lim_{x \to 0} (1 + x + 2^x - \cos x)$
7. $\lim_{x \to -2} \dfrac{x + 2}{x^2 - 4}$
8. $\lim_{x \to 0} \dfrac{x^2 - 3x}{x^3 + 4x}$
9. $\lim_{x \to 3} \dfrac{x^2 + 3x - 10}{x^2 + 5x + 6}$
10. $\lim_{x \to -2} \dfrac{2x^2 + 5x + 2}{x^2 + x - 2}$



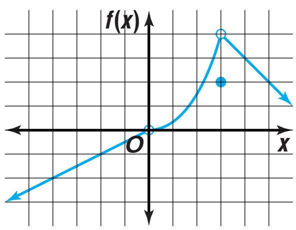
11. **Hydraulics** The velocity of a molecule of liquid flowing through a pipe depends on the distance of the molecule from the center of the pipe. The velocity, in inches per second, of a molecule is given by the function $v(r) = k(R^2 - r^2)$, where r is the distance of the molecule from the center of the pipe in inches, R is the radius of the pipe in inches, and k is a constant. Suppose for a particular liquid and a particular pipe that $k = 0.65$ and $R = 0.5$.
    a. Graph $v(r)$.
    b. Determine the limiting velocity of molecules closer and closer to the wall of the pipe.

EXERCISES

Practice

Use the graph of $y = f(x)$ to find each value.

12. $\lim_{x \to -2} f(x)$ and $f(-2)$
13. $\lim_{x \to 0} f(x)$ and $f(0)$
14. $\lim_{x \to 3} f(x)$ and $f(3)$



Evaluate each limit.

15. $\lim_{x \to 2} (-4x^2 - 3x + 6)$
16. $\lim_{x \to -1} (-x^3 + 3x^2 - 4)$
17. $\lim_{x \to \pi} \dfrac{\sin x}{x}$
18. $\lim_{x \to 0} (x + \cos x)$

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19. $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$

20. $\lim_{n \to 0} \frac{2n^2}{n}$

21. $\lim_{x \to 3} \frac{x^2 - 3x}{x^2 + 2x - 15}$

22. $\lim_{x \to 1} \frac{x^3 + 3x^2 - 4x + 8}{x + 6}$

23. $\lim_{h \to -2} \frac{h^2 + 4h + 4}{h + 2}$

24. $\lim_{x \to 3} \frac{2x^2 - 3x}{x^3 - 2x^2 + x + 6}$

25. $\lim_{x \to 0} \frac{x^3 - x^2 + 2x}{x^3 + 4x^2 - 2x}$

26. $\lim_{x \to 0} \frac{x \cos x}{x^2 + x}$

27. $\lim_{x \to 0} \frac{(x + 2)^2 - 4}{x}$

28. $\lim_{x \to -2} \frac{(x + 1)^2 - 1}{x + 2}$

29. $\lim_{x \to -2} \frac{x^3 + 8}{x^2 - 4}$

30. $\lim_{x \to 4} \frac{2x - 8}{x^3 - 64}$

31. $\lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$

32. $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$

33. Find the limit as h approaches 0 of $\frac{2h^3 - h^2 + 5h}{h}$.

34. What value does the function $g(x) = \frac{x + \pi}{\cos(x + \pi)}$ approach as x approaches 0?

Graphing Calculator

Use a graphing calculator to find the value of each limit. (Use radians with trigonometric functions.)

35. $\lim_{x \to 0} \frac{\tan 2x}{x}$

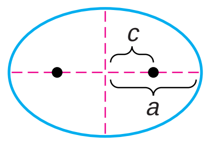
36. $\lim_{x \to 1} \frac{\ln x}{\ln(2x - 1)}$

37. $\lim_{x \to 1} \frac{1 - \sqrt{x}}{x - 1}$

38. $\lim_{x \to 0} \frac{3x - \sin 3x}{x^2 \sin x}$

Applications and Problem Solving

39. Geometry The area of an ellipse with semi-major axis a is $\pi a \sqrt{a^2 - c^2}$, where c is the distance from the foci to the center. Find the limit of the area of the ellipse as c approaches 0. Explain why the answer makes sense.



40. Biology If a population of bacteria doubles every 10 hours, then its initial hourly growth rate is $\lim_{t \to 0} \frac{2^{\frac{t}{10}} - 1}{t}$, where t is the time in hours. Use a calculator to approximate the value of this limit to the nearest hundredth. Write your answer as a percent.

41. Critical Thinking Does $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ exist? That is, can you say $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = L$ for some real number L? Explain why or why not.

42. Critical Thinking You saw in Example 5 that $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = 0.5$. That is, for values of x close to 0, $\frac{1 - \cos x}{x^2} \approx 0.5$. Solving for $\cos x$, we get $\cos x \approx 1 - \frac{x^2}{2}$.

a. Copy and complete the table by using a calculator. Round to six decimal places, if necessary.

x	1	0.5	0.1	0.01	0.001
$\cos x$					
$1 - \frac{x^2}{2}$					

b. Is it correct to say that for values of x close to 0, the expression $1 - \frac{x^2}{2}$ is a good approximation for $\cos x$? Explain.

43. Physics When an object, such as a bowling ball, is dropped near Earth's surface, the distance $d(t)$ (in feet) that the object falls in t seconds is given by $d(t) = 16t^2$. Its velocity (in feet per second) after 2 seconds is given by $\lim_{t \to 2} \frac{d(t) - d(2)}{t - 2}$. Evaluate this limit algebraically to find the velocity of the bowling ball after 2 seconds. *You will learn more about the relationship between distance and velocity in Lesson 15-2.*

44. Critical Thinking Yoshi decided that $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$ is 0, because as x approaches 0, the base of the exponential expression approaches 1, and 1 to any power is 1.

a. Use a calculator to help deduce the exact value of $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

b. Explain where Yoshi's reasoning was wrong.

Mixed Review

45. Botany A random sample of fifty acorns from an oak tree in the park reveals a mean diameter of 16.2 millimeters and a standard deviation of 1.4 millimeters. Find the range about the sample mean that gives a 99% chance that the true mean lies within it. *(Lesson 14-5)*

46. Tess is running a carnival game that involves spinning a wheel. The wheel has the numbers 1 to 10 on it. What is the probability of 7 never coming up in five spins of the wheel? *(Lesson 13-6)*

47. Find the third term of $(x - 3y)^5$. *(Lesson 12-6)*

48. Simplify $(16y^8)^{\frac{3}{4}}$. *(Lesson 11-1)*

49. Write the equation of the ellipse if the endpoints of the major axis are at $(1, -2)$ and $(9, -2)$ and the endpoints of the minor axis are at $(5, 1)$ and $(5, -5)$. *(Lesson 10-3)*

50. Graph the polar equation $r = -3$. *(Lesson 9-1)*

51. Write the ordered pair that represents $\overrightarrow{WX}$ for $W(4, 0)$ and $X(-3, -6)$. Then find the magnitude of $\overrightarrow{WX}$. *(Lesson 8-2)*

52. Transportation A car is being driven at 65 miles per hour. The car's tires have a diameter of 25 inches. What is the angular velocity of the wheels in revolutions per second? *(Lesson 6-2)*

53. Use the unit circle to find the value of csc 270°. *(Lesson 5-3)*

54. Determine the rational roots of the equation $12x^4 - 11x^3 - 54x^2 - 18x + 8 = 0$. *(Lesson 4-4)*

55. Without graphing, describe the end behavior of the function $y = 4x^5 - 2x^2 + 4$. *(Lesson 3-5)*

56. Find the value of the determinant $\begin{vmatrix} -1 & -2 \\ 3 & -6 \end{vmatrix}$. *(Lesson 2-5)*

57. Geometry Determine whether the figure with vertices at $(0, 3)$, $(8, 4)$, $(2, -5)$, and $(10, -4)$ is a parallelogram. Explain. *(Lesson 1-5)*

58. SAT Practice **Grid-In** If $2^n = 8$, what is the value of 3^{n+2}?

Extra Practice See p. A55.

GRAPHING CALCULATOR EXPLORATION

15-2A The Slope of a Curve

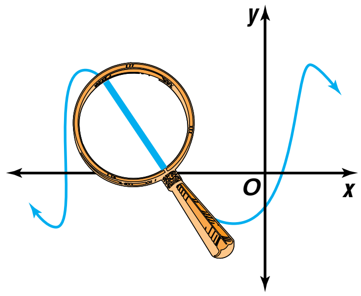
A Preview of Lesson 15-2

OBJECTIVE
- Approximate the slope of a curve.

Recall from Chapter 1 that the slope of a line is a measure of its steepness. The slope of a line is given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are the coordinates of two distinct points on the line.

What about the slope of a *curve*? A general curve does not have the same steepness at every point, but if you look at one particular point on the graph, there will be a certain steepness at that point. How would you calculate this "slope" at a particular point?

The answer lies in an important fact about curves: the graphs of most functions are "locally linear." This means that if you look at them up close, they appear to be lines. You are familiar with this phenomenon in everyday life—the surface of Earth looks flat, even though we know it is a giant sphere.

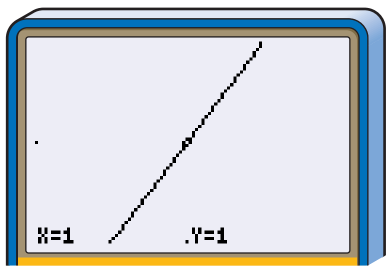


You can use ZOOM on a graphing calculator to look very closely at the graph of a function.

Example 1 **Find the slope of the graph of $y = x^2$ at (1, 1).**

Graph the equation $y = x^2$. Use the window [0, 2] by [0, 2] so that (1, 1) is at the center. Zoom in on the graph four times, using (1, 1) as the center each time. The graph should then look like the screen below. This graph is so straight that it has no visible curvature.

To approximate the slope of the graph, you can use TRACE to identify the approximate coordinates of two points on the curve. Then use the formula for slope. For example, use the coordinates (1, 1) and (1.0000831, 1.0001662).



$$m \approx \frac{1.0001662 - 1}{1.0000831 - 1}$$
$$\approx \frac{0.0001662}{0.0000831}$$
$$\approx 2$$

The slope at (1, 1) is approximately 2.

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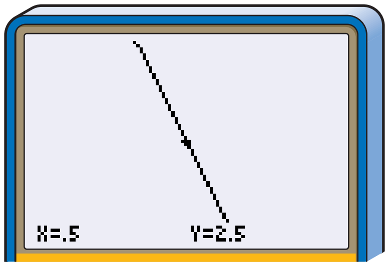
You can also have the calculator find its own approximation for the slope.

Example 2 **Find the slope of the graph of $y = \dfrac{x^2 + 1}{x}$ at (0.5, 2.5).**

Method 1: Slope Formula

Graph the equation $y = \dfrac{x^2 + 1}{x}$. Use the window [0, 1] by [2, 3] so that (0.5, 2.5) is the center. Zooming in four times results in the screen shown at the right.



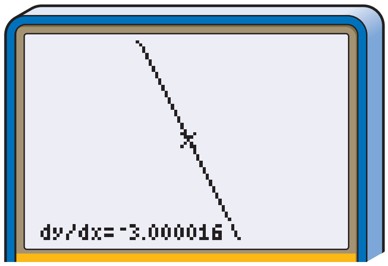
The **TRACE** feature shows that the point at (0.50004156, 2.4998753) is on the graph. Use these coordinates and (0.5, 2.5) to compute an approximate slope.

$$m \approx \frac{2.4998753 - 2.5}{0.50004156 - 0.5}$$
$$\approx -3.00048123$$

Our approximation to the slope is −3.00048123, which is quite close to −3.

Method 2: Calculator Computation

To have the calculator find an approximation, apply the ***dy/dx*** feature from the **CALC** menu at (0.5, 2.5). The calculator display is shown at the right. This also suggests that the exact value of the slope might be −3.



When you zoom in to measure the slope, you will not always obtain the exact answer. No matter how far you zoom in on the graph of a nonlinear function, the graph is never truly straight, whether it appears to be or not. Your calculation of an approximate slope may not exactly match the calculator's value for dy/dx. Sometimes your algebraic approximation may be more accurate. Other times the calculator's approximation may be more accurate.

TRY THESE

Zoom in to find the slope of the graph of each function at the given point. (Zoom in at least four times before calculating the slope.) Check your answer using the calculator's *dy/dx* feature.

1. $y = 2x^2$; (1, 2)

2. $y = \sin x$; (0, 0)

3. $y = \sqrt{x}$; (1, 1)

4. $y = 4x^4 - x^2$; (0.5, 0)

5. $y = \dfrac{1}{x - 3}$; (4, 1)

6. $y = \dfrac{x + 1}{x - 2}$; (1, −2)

WHAT DO YOU THINK?

7. For what type of function are the methods described in this lesson guaranteed to always give the exact slope?

8. What is the slope of a polynomial curve at a maximum or minimum point?

9. Graph $y = e^x$. Use the ***dy/dx*** feature to approximate the slope of the curve at several different points. What do you notice about the values of y and dy/dx?

15-2

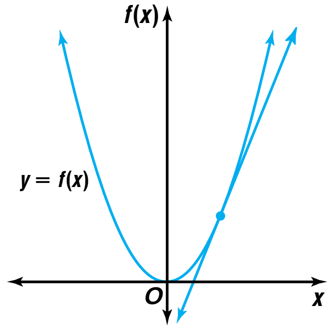
Derivatives and Antiderivatives

OBJECTIVES

- Find derivatives and antiderivatives of polynomial functions.
- Use derivatives and antiderivatives in applications.

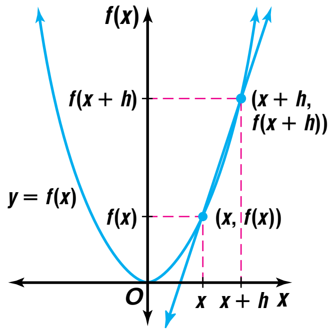
Real World Application

ROCKETRY Scott and Jabbar are testing a homemade rocket in Jabbar's back yard. The boys want to keep a record of the rocket's performance so they will know if it improves when they change the design. In physics class they learned that after the rocket uses up its fuel, the rocket's height above the ground is given by the equation $H(t) = H_0 + v_0t - 16t^2$, where H_0 is the height of the rocket (in feet) when the fuel is used up, v_0 is the rocket's velocity (in feet per second) at that time, and t is the elapsed time (in seconds) since the fuel was used up. Determine the velocity of the rocket when the fuel ran out and the maximum height the rocket reached. *This problem will be solved in Example 3.*



To solve this type of problem, we need to find the **derivative** of the function H. The derivative is related to the idea of a **tangent line** from geometry. A line tangent to a curve at a point on the curve is the line that passes through that point and has a slope equal to the slope of the curve at that point. The derivative of a function $f(x)$ is another function, $f'(x)$, that gives the slope of the tangent line to $y = f(x)$ at any point.

$f'(x)$ is read "f-prime of x."

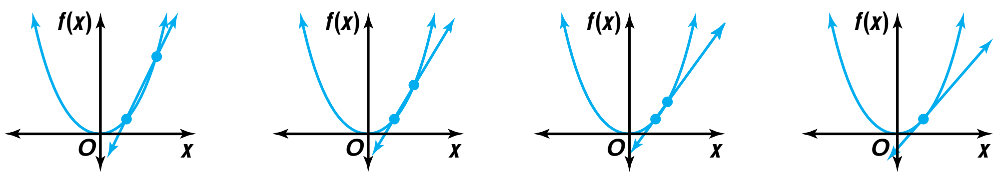


Consider the graph of $y = f(x)$ and a point $(x, f(x))$ on the graph. If the number h is close to 0, the point on the graph with x-coordinate $x + h$ will be close to $(x, f(x))$. The y-coordinate of this second point is $f(x + h)$.

Now consider the line through the points $(x, f(x))$ and $(x + h, f(x + h))$. A line that intersects a graph in two points like this is called a **secant line.** The slope of this secant line is

$$m = \frac{f(x + h) - f(x)}{(x + h) - x} \text{ or } \frac{f(x + h) - f(x)}{h}.$$

If we make h closer and closer to 0, the point $(x + h, f(x + h))$ will get closer and closer to the original point $(x, f(x))$, so the secant line will look more and more like a tangent line. This means we can compute the slope of the tangent line by finding $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$. This limit is the derivative of the function $f(x)$.

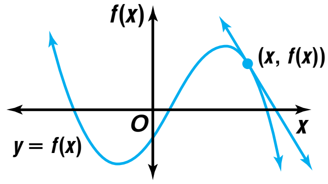
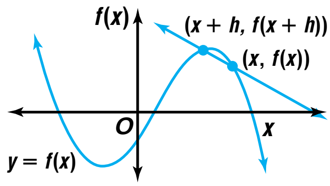


h approaches 0.

Derivative of a Function	The derivative of the function $f(x)$ is the function $f'(x)$ given by $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$\frac{dy}{dx}$ is read "dy, dx." This notation emphasizes that the derivative is a limit of slope, which is a change in y divided by a change in x.

The process of finding the derivative is called **differentiation.** Another common notation for $f'(x)$ is $\frac{dy}{dx}$. The following chart summarizes the information about tangent lines and secant lines.

Type of Line	Points of Intersection with Graph	Example	Slope
Tangent	1		$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
Secant	2		$m = \frac{f(x+h) - f(x)}{h}$

Example 1

a. Find an expression for the slope of the tangent line to the graph of $y = x^2 - 4x + 2$ at any point. That is, compute $\frac{dy}{dx}$.

b. Find the slopes of the tangent lines when $x = 0$ and $x = 3$.

a. Find and simplify $\frac{f(x+h) - f(x)}{h}$, where $f(x) = x^2 - 4x + 2$.

First, find $f(x + h)$.

$$\begin{aligned} f(x+h) &= (x+h)^2 - 4(x+h) + 2 && \textit{Replace x with x + h in f(x).} \\ &= x^2 + 2xh + h^2 - 4x - 4h + 2 \end{aligned}$$

Now find $\frac{f(x+h) - f(x)}{h}$.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{x^2 + 2xh + h^2 - 4x - 4h + 2 - (x^2 - 4x + 2)}{h} \\ &= \frac{2xh + h^2 - 4h}{h} && \textit{Simplify.} \\ &= \frac{h(2x + h - 4)}{h} && \textit{Factor.} \\ &= 2x + h - 4 && \textit{Divide by h.} \end{aligned}$$

Now find the limit of $2x + h - 4$ as h approaches 0 to compute $\frac{dy}{dx}$.

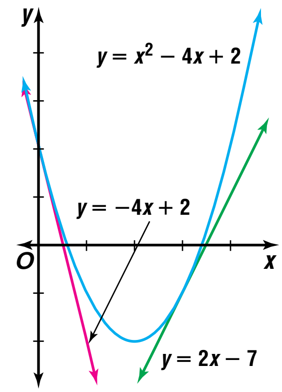
$$\begin{aligned}\frac{dy}{dx} &= f'(x)\\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h \to 0} (2x + h - 4)\\ &= 2x + 0 - 4\\ &= 2x - 4\end{aligned}$$

In the limit, only h approaches 0. x is fixed.

So $\frac{dy}{dx} = 2x - 4$.

b. At $x = 0$, $\frac{dy}{dx} = 2(0) - 4$ or -4. The slope of the tangent line at $x = 0$ is -4.

At $x = 3$, $\frac{dy}{dx} = 2(3) - 4$ or 2. The slope of the tangent line at $x = 3$ is 2.



To find the derivatives of polynomials, you can use the following rules.

Derivative Rules

Constant Rule:	The derivative of a constant function is zero. If $f(x) = c$, then $f'(x) = 0$.
Power Rule:	If $f(x) = x^n$, where n is a rational number, then $f'(x) = nx^{n-1}$.
Constant Multiple of a Power Rule:	If $f(x) = cx^n$, where c is a constant and n is a rational number, then $f'(x) = cnx^{n-1}$.
Sum and Difference Rule:	If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$.

Example 2 **Find the derivative of each function.**

a. $f(x) = x^6$

$$\begin{aligned}f'(x) &= 6x^{6-1} && \textit{Power Rule}\\ &= 6x^5\end{aligned}$$

b. $f(x) = x^2 - 4x + 2$

$$\begin{aligned}f(x) &= x^2 - 4x + 2\\ &= x^2 - 4x^1 + 2 && \textit{Rewrite x as a power.}\end{aligned}$$

$$\begin{aligned}f'(x) &= 2x^{2-1} - 4 \cdot 1x^{1-1} + 0 && \textit{Use all four rules.}\\ &= 2x^1 - 4x^0\\ &= 2x - 4 && x^0 = 1\end{aligned}$$

c. $f(x) = 2x^4 - 7x^3 + 12x^2 - 8x - 10$

$$\begin{aligned}f'(x) &= 2 \cdot 4x^3 - 7 \cdot 3x^2 + 12 \cdot 2x - 8 \cdot 1 - 0\\ &= 8x^3 - 21x^2 + 24x - 8\end{aligned}$$

d. $f(x) = x^3 (x^2 + 5)$

$$\begin{aligned} f(x) &= x^3 (x^2 + 5) \\ &= x^5 + 5x^3 \quad \textit{Multiply to write the function as a polynomial.} \end{aligned}$$

$$\begin{aligned} f'(x) &= 5x^4 + 5 \cdot 3x^2 \\ &= 5x^4 + 15x^2 \end{aligned}$$

e. $f(x) = (x^2 + 4)^2$

$$\begin{aligned} f(x) &= (x^2 + 4)^2 \\ &= x^4 + 8x^2 + 16 \quad \textit{Square to write the function as a polynomial.} \end{aligned}$$

$$\begin{aligned} f'(x) &= 4x^3 + 8 \cdot 2x + 0 \\ &= 4x^3 + 16x \end{aligned}$$

Suppose $s(t)$ is the displacement of a moving object at time t. For example, $s(t)$ might be the object's altitude or its distance from its starting point. Then the derivative, denoted $s'(t)$ or $\frac{ds}{dt}$, is the velocity of the object at time t. Velocity is usually denoted by $v(t)$.

Example 3 Real World Application

ROCKETRY **Refer to the application at the beginning of the lesson. Suppose Scott's stopwatch shows that the rocket reached its highest point 5.3 seconds after its fuel was exhausted. Jabbar's stopwatch says that the rocket hit the ground 12.7 seconds after the fuel ran out.**

a. How fast was the rocket moving at the instant its fuel ran out?

b. What was the maximum height of the rocket?

a. We have to find the value of v_0. This value cannot be found directly from the height function $H(t)$ because H_0 is still unknown. Instead we use the velocity function $v(t)$ and what we can deduce about the velocity of the rocket at its highest point.

$$\begin{aligned} H(t) &= H_0 + v_0 t - 16t^2 \\ v(t) &= H'(t) \quad \textit{The velocity of the rocket is the derivative of its height.} \\ &= 0 + v_0 \cdot 1 - 16 \cdot 2t \quad \textit{H_0 and v_0 are constants; t is the variable.} \\ &= v_0 - 32t \end{aligned}$$

When the rocket was at its highest point, it was neither rising nor falling, so its velocity was 0. Substituting $v(t) = 0$ and $t = 5.3$ into the equation $v(t) = v_0 - 32t$ yields $0 = v_0 - 32(5.3)$, or $v_0 = 169.6$.

The velocity of the rocket was 169.6 ft/s when the fuel ran out.

b. We can now write the equation for the height of the rocket as $H(t) = H_0 + 169.6t - 16t^2$. When the rocket hit the ground, its height $H(t)$ was 0, so we substitute $H(t) = 0$ and $t = 12.7$ into the height equation.

$$\begin{aligned} H(t) &= H_0 + 169.6t - 16t^2 \\ 0 &= H_0 + 169.6(12.7) - 16(12.7)^2 \quad \textit{$H(t) = 0$, $t = 12.7$} \\ 16(12.7)^2 - 169.6(12.7) &= H_0 \quad \textit{Solve for H_0.} \\ H_0 &= 426.72 \end{aligned}$$

The height of the rocket can now be written as $H(t) = 426.72 + 169.6t - 16t^2$. To find the maximum height of the rocket, which occurred at $t = 5.3$, compute $H(5.3)$.

$$H(t) = 426.72 + 169.6t - 16t^2$$
$$H(5.3) = 426.72 + 169.6(5.3) - 16(5.3)^2 \quad \text{Replace } t \text{ with } 5.3.$$
$$= 876.16$$

The maximum height of the rocket was about 876 feet.

Finding the **antiderivative** of a function is the inverse of finding the derivative. That is, instead of finding the derivative of $f(x)$, you are trying to find a function whose derivative is $f(x)$. For a function $f(x)$, the antiderivative is often denoted by $F(x)$. The relationship between the two functions is $F'(x) = f(x)$.

Example 4 **Find the antiderivative of the function $f(x) = 2x$.**

We are looking for a function whose derivative is $2x$. You may recall from previous examples that the function x^2 fits that description. The derivative of x^2 is $2x^{2-1}$, or $2x$.

However, x^2 is not the only function that works. The function $G(x) = x^2 + 1$ is another, since its derivative is $G'(x) = 2x + 0$ or $2x$. Another answer is $H(x) = x^2 + 17$, and still another is $J(x) = x^2 - 6$. In fact, adding any constant, positive or negative, to x^2 does not change the fact that the derivative is $2x$.

So there is an endless list of answers, all of which can be summarized by the expression $x^2 + C$, where C is any constant. So for the function $f(x) = 2x$, we say the antiderivative is $F(x) = x^2 + C$.

As with derivatives, there are rules for finding antiderivatives.

Antiderivative Rules

Power Rule:	If $f(x) = x^n$, where n is a rational number other than -1, the antiderivative is $F(x) = \frac{1}{n+1}x^{n+1} + C$.
Constant Multiple of a Power Rule:	If $f(x) = kx^n$, where n is a rational number other than -1 and k is a constant, the antiderivative is $F(x) = k \cdot \frac{1}{n+1}x^{n+1} + C$.
Sum and Difference Rule:	If the antiderivatives of $f(x)$ and $g(x)$ are $F(x)$ and $G(x)$, respectively, then the antiderivative of $f(x) \pm g(x)$ is $F(x) \pm G(x)$.

Example 5 **Find the antiderivative of each function.**

a. $f(x) = 3x^7$

$F(x) = 3 \cdot \frac{1}{7+1}x^{7+1} + C$ *Constant Multiple of a Power Rule*

$= \frac{3}{8}x^8 + C$

b. $f(x) = 4x^2 - 7x + 5$

$f(x) = 4x^2 - 7x + 5$

$= 4x^2 - 7x^1 + 5x^0$ *Rewrite the function so that each term has a power of x.*

$F(x) = 4 \cdot \frac{1}{3}x^3 + C_1 - \left(7 \cdot \frac{1}{2}x^2 + C_2\right) + 5 \cdot \frac{1}{1}x^1 + C_3$ *Constant Multiple of a Power and Sum and Difference Rules*

$= \frac{4}{3}x^3 - \frac{7}{2}x^2 + 5x + C$ *Let $C = C_1 - C_2 + C_3$.*

c. $f(x) = x(x^2 + 2)$

$f(x) = x(x^2 + 2)$

$= x^3 + 2x$ *Multiply to write the function as a polynomial.*

$F(x) = \frac{1}{4}x^4 + C_1 + 2 \cdot \frac{1}{2}x^2 + C_2$ *Use all three antiderivative rules.*

$= \frac{1}{4}x^4 + x^2 + C$ *Let $C = C_1 + C_2$.*

In real-world situations, the derivative of a function is often called the **rate of change** of the function because it measures how fast the function changes. If you are given the derivative or rate of change of a function, you can find the antiderivative to recover the original function. If given additional information, you may also be able to find a value for the constant *C*.

Example 6 Real World Application

CENSUS **Data on the growth of world population provided by the U. S. Census Bureau can be used to create a model of Earth's population growth. According to this model, the rate of change of the world's population since 1950 is given by $p(t) = -0.012t^2 + 48t - 47{,}925$, where t is the calendar year and $p(t)$ is in millions of people per year.**

a. Given that the population in 2000 was about 6000 million people, find an equation for $P(t)$, the total population as a function of the calendar year.

b. Use the equation from part a to predict the world population in 2050.

a. $P(t)$ is the antiderivative of $p(t)$.

$p(t) = -0.012t^2 + 48t - 47{,}925$

$P(t) = -0.012 \cdot \frac{1}{3}t^3 + 48 \cdot \frac{1}{2}t^2 - 47{,}925t + C$ *Antiderivative rules*

$= -0.004t^3 + 24t^2 - 47{,}925t + C$

interNET CONNECTION

Data Update For the latest information about the population of the U.S. and the world, visit: **www.amc.glencoe.com**

To find C, substitute 2000 for t and 6000 for $P(t)$.

$$6000 = -0.004(2000)^3 + 24(2000)^2 - 47{,}925(2000) + C$$
$$6000 = -32{,}000{,}000 + 96{,}000{,}000 - 95{,}850{,}000 + C$$
$$C = 31{,}856{,}000 \quad \textit{Solve for C.}$$

Substituting this value of C into our formula for $P(t)$ gives $P(t) = -0.004t^3 + 24t^2 - 47{,}925t + 31{,}856{,}000$. Of all the antiderivatives of $p(t)$, this is the only one that gives the proper population for the year 2000.

b. Substitute 2050 for t.

$$P(t) = -0.004t^3 + 24t^2 - 47{,}925t + 31{,}856{,}000$$
$$P(2050) = -0.004(2050)^3 + 24(2050)^2 - 47{,}925(2050) + 31{,}856{,}000$$
$$= 9250$$

According to the model, the world population in 2050 should be about 9250 million, or 9.25 billion.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** two different sentences that describe the relationship between the functions $4x^3$ and x^4, one using the word *derivative,* the other using the word *antiderivative.*
2. **Explain** why the Power Rule for antiderivatives is not valid when $n = -1$.
3. *Math Journal* **Write** a paragraph explaining the difference between $f(x + h)$ and $f(x) + h$. What answer would you always get if you mistakenly used $f(x) + h$ when finding a derivative using the definition?

Guided Practice

Use the definition of derivative to find the derivative of each function.

4. $f(x) = 3x + 2$
5. $f(x) = x^2 + x$

Use the derivative rules to find the derivative of each function.

6. $f(x) = 2x^2 - 3x + 5$
7. $f(x) = -x^3 - 2x^2 + 3x + 6$
8. $f(x) = 3x^4 + 2x^3 - 3x - 2$
9. Find the slope of the tangent line to the graph of $y = x^2 + 2x + 3$ at the point where $x = 1$.

Find the antiderivative of each function.

10. $f(x) = x^2$
11. $f(x) = x^3 + 4x^2 - x - 3$
12. $f(x) = 5x^5 + 2x^3 - x^2 + 4$

13. **Business** The Better Book Company finds that the cost, in dollars, to print x copies of a book is given by the function $C(x) = 1000 + 10x - 0.001x^2$. The derivative $C'(x)$ is called the *marginal cost function*. The marginal cost is the approximate cost of printing one more book after x copies have been printed. What is the marginal cost when 1000 books have been printed?

EXERCISES

Practice

Use the definition of derivative to find the derivative of each function.

14. $f(x) = 2x$
15. $f(x) = 7x + 4$
16. $f(x) = -3x$
17. $f(x) = -4x - 9$
18. $f(x) = 2x^2 + 5x$
19. $f(x) = x^3 + 5x^2 + 6$

Use the derivative rules to find the derivative of each function.

20. $f(x) = 8x$
21. $f(x) = 2x + 6$
22. $f(x) = \frac{1}{3}x + \frac{4}{5}$
23. $f(x) = -3x^2 + 2x + 9$
24. $f(x) = \frac{1}{2}x^2 - x - 2$
25. $f(x) = x^3 - 2x^2 + 5x - 6$
26. $f(x) = 3x^4 + 7x^3 - 2x^2 + 7x - 12$
27. $f(x) = (x^2 + 3)(2x - 7)$
28. $f(x) = (2x + 4)^2$
29. $f(x) = (3x - 4)^3$
30. Find $f'(x)$ for the function $f(x) = \frac{2}{3}x^3 + \frac{1}{3}x^2 - x - 9$.

Find the slope of the tangent line to the graph of each equation at $x = 1$.

31. $y = x^3$
32. $y = x^3 - 7x^2 + 4x + 9$
33. $y = (x + 1)(x - 2)$
34. $y = (5x^2 + 7)^2$

Find the antiderivative of each function.

35. $f(x) = x^6$
36. $f(x) = 3x + 4$
37. $f(x) = 4x^2 - 6x + 7$
38. $f(x) = 12x^2 - 6x + 1$
39. $f(x) = 8x^3 + 5x^2 - 9x + 3$
40. $f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^2 + 4$
41. $f(x) = (2x + 3)(3x - 7)$
42. $f(x) = x^4(x + 2)^2$
43. $f(x) = \frac{x^3 + 4x^2 + x}{x}$
44. $f(x) = \frac{2x^2 - 5x - 3}{x - 3}$
45. Find a function whose derivative is $f(x) = (x^3 - 1)(x^2 + 1)$.

Applications and Problem Solving

Real World Application

46. **Motion** Acceleration is the rate at which the velocity of a moving object changes. That is, acceleration is the derivative of velocity. If time is measured in seconds and velocity in feet per second, then acceleration is measured in feet per second squared, or ft/s^2. Suppose a car is moving with velocity $v(t) = 15 + 4t + \frac{1}{8}t^2$. *Feet per second squared is feet per second per second.*
    a. Find the car's velocity at $t = 12$.
    b. Find the car's acceleration at $t = 12$.
    c. Interpret your answer to part b in words.
    d. Suppose $s(t)$ is the car's distance, in feet, from its starting point. Find an equation for $s(t)$.
    e. Find the distance the car travels in the first 12 seconds.

www.amc.glencoe.com/self_check_quiz

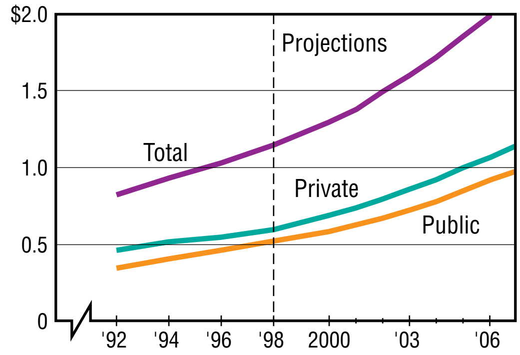
47. Critical Thinking Use the definition of derivative to find the derivative of $f(x) = \frac{1}{x}$.

48. Economics The graph shows the annual spending on health care in the U.S. for the years 1992 to 2006 (using projections for the years after 1998.) Let $T(y)$ be the total annual spending on health care in year y.

a. Estimate $T(2003)$ and describe what it measures.

b. Estimate $T'(2003)$ and describe what it measures.

Health Care Spending (Trillions of Dollars)



Source: Health Care Financing Administration

49. Sports Suppose a punter kicks a football so that the upward component of its velocity is 80 feet per second. If the ball is 3 feet off the ground when it is kicked, then the height of the ball, in feet, t seconds after it is kicked is given by $h(t) = 3 + 80t - 16t^2$.

a. Find the upward velocity $v(t)$ of the football.

b. How fast is the ball travelling upward 1 second after it is kicked?

c. Find the time when the ball reaches its maximum height.

d. What is the maximum height of the ball?

50. Critical Thinking The derivative of the function $f(x) = e^x$ is *not* xe^{x-1}. (e^x is an exponential function, so the Power Rule for derivatives does not apply.) Use the definition of derivative to find the correct derivative. (*Hint*: You will need a calculator to evaluate a limit that arises in the computation.)

51. Business Joaquin and Marva are selling lemonade. The higher the price they charge for a cup of lemonade, the fewer cups they sell. They have found that when they charge p cents for a cup of lemonade, they sell $100 - 2p$ cups in a day.

a. Find a formula for the function $r(p)$ that gives their total daily revenue.

b. Find the price that Joaquin and Marva should charge to generate the highest possible revenue.

Mixed Review

52. Evaluate $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$. *(Lesson 15-1)*

53. Nutrition The amounts of sodium, in milligrams, present in the top brands of peanut butter are given below. *(Lesson 14-3)*

195	210	180	225	225	225	195
225	203	225	195	195	188	191
210	233	225	248	225	210	240
180	225	240	180	225	240	240
195	189	178	255	225	225	225
194	210	225	195	188	205	

a. Make a box-and-whisker plot of the data.

b. Write a paragraph describing the variability of the data.

54. A pair of dice is tossed. Find the probability that their sum is greater than 7 given that the numbers match. *(Lesson 13-5)*

55. The first term of a geometric sequence is 9, and the common ratio is $-\frac{1}{3}$. Find the sixth term of the sequence. *(Lesson 12-2)*

56. **Chemistry** A beaker of water has been heated to 210°F in a room that is 74°F. Use Newton's Law of Cooling, $y = ae^{-kt} + c$, with $a = 136$°F, $k = 0.06$ min^{-1}, and $c = 74$°F to find the temperature of the water after half an hour. *(Lesson 11-3)*

57. Write the standard form of the equation of the circle that passes through points at $(2, -1)$, $(-3, 0)$, and $(1, 4)$. *(Lesson 10-2)*

58. Express $5\left(\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}\right)$ in rectangular form. *(Lesson 9-6)*

59. Write parametric equations of the line passing through $P(-3, -2)$ and parallel to $\vec{\mathbf{v}} = \langle 8, 3 \rangle$. *(Lesson 8-6)*

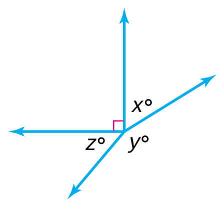
60. Graph $y = -3 \sin(\theta - 45°)$. *(Lesson 6-5)*

61. **Surveying** A surveying crew is studying a housing project for possible relocation for the airport expansion. They are located on the ground, level with the houses. If the distance to one of the houses is 253 meters and the distance to the other is 319 meters, what is the distance between the houses if the angle subtended by them at the point of observation is 42°12′? *(Lesson 5-8)*

62. List the possible rational roots of $2x^3 + 3x^2 - 8x + 3 = 0$. Then determine the rational roots. *(Lesson 4-4)*

63. **SAT/ACT Practice**
In the figure, $x + y + z = ?$

A 0 **B** 90 **C** 180
D 270 **E** 360



MID-CHAPTER QUIZ

Evaluate each limit. (Lesson 15-1)

1. $\lim_{x \to -3} (2x^2 - 4x + 6)$

2. $\lim_{x \to 2} \frac{x^2 - 9x + 14}{2x^2 - 7x + 6}$

3. $\lim_{x \to 0} \frac{\sin 2x}{x}$

4. Use the definition of derivative to find the derivative of $f(x) = x^2 - 3$. (Lesson 15-2)

Use the derivative rules to find the derivative of each function. (Lesson 15-2)

5. $f(x) = \pi$

6. $f(x) = 3x^2 - 5x + 2$

7. **Medicine** If $R(M)$ measures the reaction of the body to an amount M of medicine, then $R'(M)$ measures the sensitivity of the body to the medicine. Find $R'(M)$ if $R(M) = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$ where C is a constant.

Find the antiderivative of each function. (Lesson 15-2)

8. $f(x) = -x^2 + 7x - 6$

9. $f(x) = 2x^3 + x^2 + 8$

10. $f(x) = -2x^4 + 6x^3 - 2x - 5$

Extra Practice See p. A55.

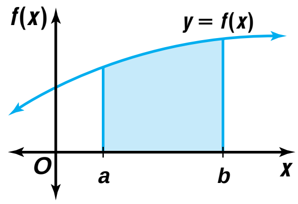
15-3 Area Under a Curve

OBJECTIVES
- Find values of integrals of polynomial functions.
- Find areas under graphs of polynomial functions.

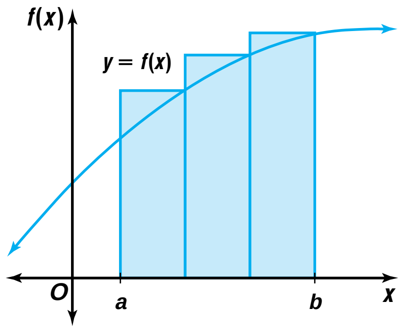
Real World Application

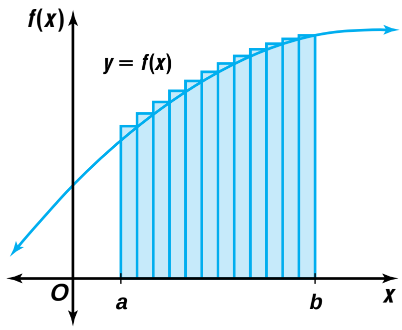
BUSINESS The derivative of a cost function is called a *marginal cost function.* A shoe company determines that the marginal cost function for a particular type of shoe is $f(x) = 20 - 0.004x$, where x is the number of pairs of shoes manufactured and $f(x)$ is in dollars. If the company is already producing 2000 pairs of this type of shoe per day, how much more would it cost them to increase production to 3000 pairs per day? *This problem will be solved in Example 3.*

Problems like the one above can be solved using **integrals.** To understand integrals, we must first examine the area between the graph of a polynomial function and the x-axis for an interval from $x = a$ to $x = b$.



One way to estimate this area is by filling the region with rectangles, whose areas we know how to compute. If the boundary of the region is curved, the rectangles will not fit the region exactly, but you can use them for approximation. You can use rectangles of any width.



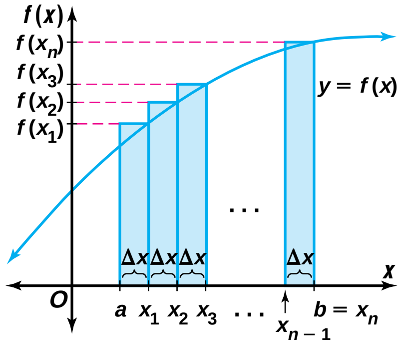


interNET CONNECTION
Graphing Calculator Programs
To download a program that uses rectangles to approximate the area under a curve, visit: **www.amc.glencoe.com**

Notice from the figures above that the thinner the rectangles are, the better they fit the region, and the better their total area approximates the area of the region. If you were to continue making the rectangles thinner and thinner, their total area would approach the exact area of the region. That is, the area of a region under the graph of a function is the limit of the total area of the rectangles as the widths of the rectangles approach 0.

In the figure below, the interval from a to b has been subdivided into n equal subintervals. A rectangle has been drawn on each subinterval. Each rectangle touches the graph at its upper right corner; the first touches at the x-coordinate x_1, the second touches at the x-coordinate x_2, and so on, with the last rectangle touching at the x-coordinate b, which is also denoted by x_n for consistency.

The height of the first rectangle is $f(x_1)$, the height of the second is $f(x_2)$, and so on, with the height of the last rectangle being $f(x_n)$. The length of the entire interval from a to b is $b - a$, so the width of each of the n rectangles must be $\frac{b-a}{n}$. This common width is traditionally denoted Δx. *Δx is read "delta x."*



Look Back
You can refer to Lesson 12-5 to review sigma notation.

The area of the first rectangle is $f(x_1)\Delta x$, the area of the second rectangle is $f(x_2)\Delta x$, and so on. The total area A_n of the n rectangles is given by the sum of the areas.

$$A_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$ *i is the index of summation, not the imaginary unit.*

$\int_a^b f(x)\,dx$ is read "the integral of f(x) from a to b."

To make the width of the rectangles approach 0, we let the number of rectangles approach infinity. Therefore, the exact area of the region under the graph of the function is $\lim_{n\to\infty} A_n$, or $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. This limit is called a **definite integral** and is denoted $\int_a^b f(x)\,dx$.

Definite Integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \text{ where } \Delta x = \frac{b-a}{n}.$$

The process of finding the area under a curve is called **integration.** The following formulas will be needed in the examples and exercises.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

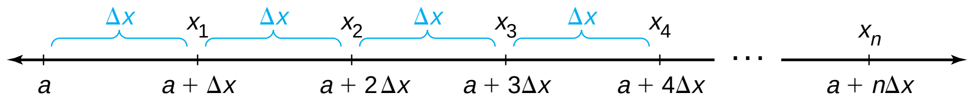
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

$$1^5 + 2^5 + 3^5 + \cdots + n^5 = \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12}$$

Before beginning the examples, we will derive a formula for x_i. The width Δx of each rectangle is the distance between successive x_i-values. Study the labels below the x-axis.

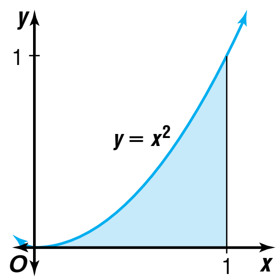


We see that $x_i = a + i\Delta x$. This formula will work when finding the area under the graph of any function.

Example 1 **Use limits to find the area of the region between the graph of $y = x^2$ and the x-axis from $x = 0$ to $x = 1$. That is, find $\int_0^1 x^2\, dx$.**

First find Δx.

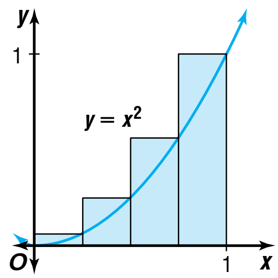
$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-0}{n} \text{ or } \frac{1}{n}$$



Then find x_i.

$$x_i = a + i\Delta x$$
$$= 0 + i \cdot \frac{1}{n} \text{ or } \frac{i}{n}$$

Now we can calculate the integral that gives the area.



$$\int_0^1 x^2\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^2 \Delta x \qquad f(x_i) = x_i^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \qquad x_i = \frac{i}{n},\ \Delta x = \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} \qquad \textit{Multiply.}$$

$$= \lim_{n\to\infty} \left(\frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{n^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \qquad \textit{Factor.}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \qquad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} \qquad \textit{Multiply.}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \qquad \textit{Factor and divide by } n^2.$$

$$= \left(\lim_{n\to\infty} \frac{1}{6}\right)\left[\lim_{n\to\infty} 2 + \left(\lim_{n\to\infty} 3\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) + \lim_{n\to\infty} \frac{1}{n^2}\right] \qquad \textit{Limit theorems from Chapter 12}$$

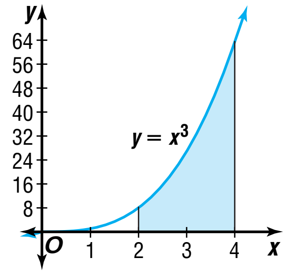
$$= \frac{1}{6}\left[2 + (3)(0) + 0\right] \text{ or } \frac{1}{3} \qquad \lim_{n\to\infty} \frac{1}{n} = 0,\ \lim_{n\to\infty} \frac{1}{n^2} = 0$$

The area of the region is $\frac{1}{3}$ square unit.

Example 2 **Use limits to find the area of the region between the graph of $y = x^3$ and the x-axis from $x = 2$ to $x = 4$.**

First, find the area under the graph from $x = 0$ to $x = 4$. Then subtract from it the area under the graph from $x = 0$ to $x = 2$. In other words,

$$\int_2^4 x^3\,dx = \int_0^4 x^3\,dx - \int_0^2 x^3\,dx.$$



For $\int_0^4 x^3\,dx$, $a = 0$ and $b = 4$, so $\Delta x = \frac{4}{n}$ and $x_i = \frac{4i}{n}$.

$$\int_0^4 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^3\,\Delta x \qquad f(x_i) = x_i^3$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^3 \cdot \frac{4}{n} \qquad x_i = \frac{4i}{n},\ \Delta x = \frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{256i^3}{n^4}$$

$$= \lim_{n\to\infty} \left(\frac{256 \cdot 1^3}{n^4} + \frac{256 \cdot 2^3}{n^4} + \cdots + \frac{256 \cdot n^3}{n^4}\right)$$

$$= \lim_{n\to\infty} \frac{256}{n^4} \cdot (1^3 + 2^3 + \cdots + n^3)$$

$$= \lim_{n\to\infty} \frac{256}{n^4} \cdot \frac{n^2(n+1)^2}{4} \qquad 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{64n^2 + 128n + 64}{n^2}$$

$$= \lim_{n\to\infty} \left(64 + \frac{128}{n} + \frac{64}{n^2}\right) \qquad \text{Divide by } n^2.$$

$$= 64 + 0 + 0 \text{ or } 64$$

For $\int_0^2 x^3\,dx$, $a = 0$ and $b = 2$, so $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$.

$$\int_0^2 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^3\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \cdot \frac{2}{n} \qquad x_i = \frac{2i}{n},\ \Delta x = \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i^3}{n^4}$$

$$= \lim_{n\to\infty} \left(\frac{16 \cdot 1^3}{n^4} + \frac{16 \cdot 2^3}{n^4} + \cdots + \frac{16 \cdot n^3}{n^4}\right)$$

$$= \lim_{n\to\infty} \frac{16}{n^4} \cdot (1^3 + 2^3 + \cdots + n^3)$$

$$= \lim_{n\to\infty} \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \qquad 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{4n^2 + 8n + 4}{n^2}$$

$$= \lim_{n\to\infty}\left(4 + \frac{8}{n} + \frac{4}{n^2}\right) \quad \text{Divide by } n^2.$$

$$= 4 + 0 + 0 \text{ or } 4$$

The area of the region between the graph of $y = x^3$ and the x-axis from $x = 2$ to $x = 4$ is $64 - 4$, or 60 square units.

In physics, when the velocity of an object is graphed with respect to time, the area under the curve represents the displacement of the object. In business, the area under the graph of a marginal cost function from $x = a$ to $x = b$ represents the amount it would cost to increase production from a units to b units.

Example 3 Real World Application

Since $f(x)$ is a linear function, we can calculate the value directly, without subtracting integrals as in Example 2.

BUSINESS Refer to the application at the beginning of the lesson. How much would it cost the shoe company to increase production from 2000 pairs per day to 3000 pairs per day?

The cost is given by $\int_{2000}^{3000} f(x)\,dx$ where $f(x) = 20 - 0.004x$ is the marginal cost function.

$a = 2000$ and $b = 3000$, so $\Delta x = \frac{1000}{n}$ and $x_i = 2000 + \frac{1000i}{n}$.

$$\int_{2000}^{3000} f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(20 - 0.004x_i)\Delta x \quad f(x_i) = 20 - 0.004x_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[20 - 0.004\left(2000 + \frac{1000i}{n}\right)\right]\cdot\frac{1000}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(12 - \frac{4i}{n}\right)\cdot\frac{1000}{n} \quad \text{Simplify.}$$

$$= \lim_{n\to\infty}\frac{1000}{n}\left[\left(12 - \frac{4\cdot 1}{n}\right) + \left(12 - \frac{4\cdot 2}{n}\right) + \cdots + \left(12 - \frac{4\cdot n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1000}{n}\cdot\left[12n - \frac{4}{n}(1 + 2 + \cdots + n)\right] \quad \text{Combine and factor.}$$

$$= \lim_{n\to\infty}\frac{1000}{n}\cdot\left[12n - \frac{4}{n}\cdot\frac{n(n+1)}{2}\right] \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{1000}{n}\cdot(10n - 2) \quad \text{Simplify.}$$

$$= \lim_{n\to\infty}\frac{10{,}000n - 2000}{n} \quad \text{Multiply.}$$

$$= \lim_{n\to\infty}\left(10{,}000 - \frac{2000}{n}\right) \quad \text{Divide by } n.$$

$$= 10{,}000 - 0, \text{ or } 10{,}000$$

The increase in production would cost the company \$10,000.

Check for Understanding

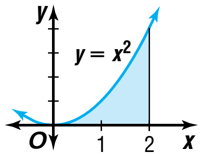
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** an equation of a function for which you would need the formula for $1^4 + 2^4 + 3^4 + \cdots + n^4$ to find the area under the graph.
2. **Describe** the steps involved in finding the area under the graph of $y = f(x)$ between $x = a$ and $x = b$.
3. **You Decide** Rita says that when you use rectangles that touch the graph of a function at their upper right corners, the total area of the rectangles will always be greater than the area under the curve because the rectangles stick out above the curve. Lorena disagrees. Who is correct? Explain.

Guided Practice

4. Use a limit to find the area of the shaded region in the graph at the right.



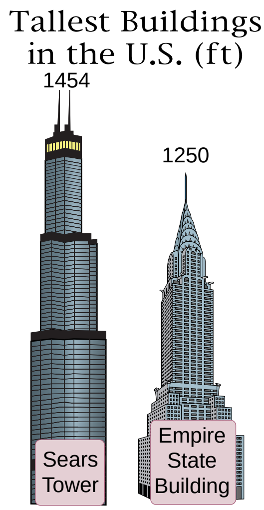
Use limits to find the area between each curve and the *x*-axis for the given interval.

5. $y = x^2$ from $x = 1$ to $x = 3$
6. $y = x^3$ from $x = 0$ to $x = 1$

Use limits to evaluate each integral.

7. $\int_0^6 x^2\,dx$
8. $\int_0^3 x^3\,dx$

9. **Physics** Neglecting air resistance, an object in free fall accelerates at 32 feet per second squared. So the velocity of the object t seconds after being dropped is $32t$ feet per second. Suppose a ball is dropped from the top of the Sears Tower.
   a. Use integration to find how far the ball would fall in the first six seconds.
   b. Refer to the graph at the right. Would the ball hit the ground within ten seconds of being dropped? Explain your reasoning.

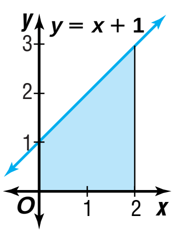


Exercises

Practice

Use limits to find the area of the shaded region in each graph.

10.



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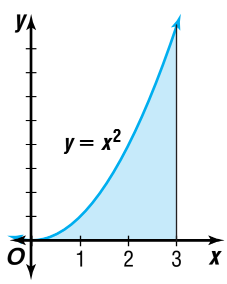

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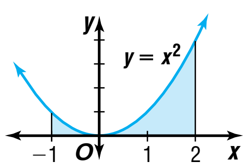

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CONTENTS

Use limits to find the area between each curve and the *x*-axis for the given interval.

13. $y = x$ from $x = 1$ to $x = 3$

14. $y = x^2$ from $x = 0$ to $x = 5$

15. $y = 2x^3$ from $x = 1$ to $x = 5$

16. $y = x^4$ from $x = 0$ to $x = 5$

17. $y = x^2 + 6x$ from $x = 0$ to $x = 4$

18. $y = x^2 - x + 1$ from $x = 0$ to $x = 3$

19. Write a limit that gives the area under the graph of $y = \sin x$ from $x = 0$ to $x = \pi$. (Do *not* evaluate the limit.)

Use limits to evaluate each integral.

20. $\int_0^2 8x\,dx$

21. $\int_1^4 (x + 2)\,dx$

22. $\int_0^4 x^2\,dx$

23. $\int_3^5 8x^3\,dx$

24. $\int_1^4 (x^2 + 4x - 2)\,dx$

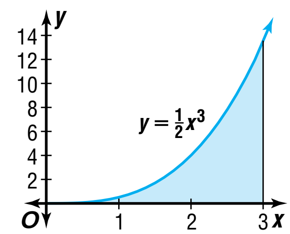
25. $\int_0^2 (x^5 + x^2)\,dx$

26. Find the integral of x^3 from 0 to 5.

Applications and Problem Solving

Real World Application

27. Sewing A patch in the shape of the region shown at the right is to be sewn onto a flag. If each unit in the coordinate system represents one foot, how much material is required for the patch?

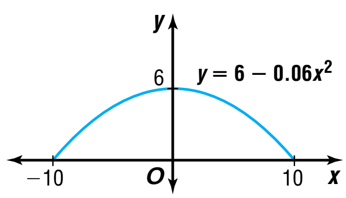


28. Business Suppose the Auburn Widget Corporation finds that the marginal cost function associated with producing x widgets is $f(x) = 80 - 2x$ dollars.

a. Refer to Exercise 13 of Lesson 15-2. Use the marginal cost function to approximate the cost for the company to produce one more widget when the production level is 20 widgets.

b. How much would it cost the company to double its production from 20 widgets to 40 widgets?

29. Mining In order to distribute stress, mine tunnels are sometimes rounded. Suppose that the vertical cross sections of a tunnel can be modeled by the parabola $y = 6 - 0.06x^2$. If x and y are measured in feet, how much rock would have to be moved to make such a tunnel that is 100 feet long?



30. Critical Thinking Find the area of the region enclosed by the line $y = x$ and the parabola $y = x^2$.

31. **Budgets** If the function $r(t)$ gives the rate at which a family spends money, then the total money spent between times $t = a$ and $t = b$ is $\int_a^b r(t)\,dt$. A local electric company in Alabama, where electric bills are generally low in winter and very high in summer, offers customers the option of paying a flat monthly fee for electricity throughout the year so that customers can avoid enormous summertime bills. The company has found that in past years the Johnson family's rate of electricity spending can be modeled by $r(t) = 50 + 36t - 3t^2$ dollars per month, where t is the number of months since the beginning of the year.

a. Sketch a graph of the function $r(t)$ for $0 \le t \le 12$.

b. Find the total amount of money the Johnsons would spend on electricity during a full year.

c. If the Johnsons choose the option of paying a flat monthly fee, how much should the electric company charge them each month?

32. **Sports** A sprinter is trying to decide between two strategies for running a race. She can put a lot of energy into an initial burst of speed, which gives her a velocity of $v(t) = 3.5t - 0.25t^2$ meters per second after t seconds, or she can save her energy for more acceleration at the end so that her velocity is given by $v(t) = 1.2t + 0.03t^2$.

a. Graph the two velocity functions on the same set of axes for $0 \le t \le 10$.

b. Use integration to determine which velocity results in a greater distance covered in a 10-second race.

33. **Critical Thinking** Find the value of $\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$, where r is a constant.

Mixed Review

34. Find the derivative of $f(x) = -3x^3 + x^2 - 7x$. *(Lesson 15-2)*

35. Evaluate $\lim_{x \to 2} \frac{x - 2}{x + 2}$. *(Lesson 15-1)*

36. Solve the equation $\log_{\frac{1}{3}} x = -3$. *(Lesson 11-4)*

37. Find an ordered triple to represent $\vec{\mathbf{u}}$ if $\vec{\mathbf{u}} = \vec{\mathbf{v}} + \vec{\mathbf{w}}$, $\vec{\mathbf{v}} = \langle 2, -5, -3 \rangle$, and $\vec{\mathbf{w}} = \langle -3, 4, -7 \rangle$. Then write $\vec{\mathbf{u}}$ as the sum of unit vectors. *(Lesson 8-3)*

38. If $\sin r = \frac{3}{5}$ and r is in the first quadrant, find $\cos 2r$. *(Lesson 7-4)*

39. State the amplitude and period for the function $y = \frac{1}{2} \sin 10\theta$. *(Lesson 6-4)*

40. **Manufacturing** A cereal manufacturer wants to make a cardboard cereal box of maximum volume. The function representing the volume of the box is $v(x) = -0.7x^3 + 5x^2 + 7x$, where x is the width of the box in centimeters. Find the width of the box that will maximize the volume. *(Lesson 3-6)*

41. **SAT/ACT Practice** Triangle ABC has sides that are 6, 8, and 10 inches long. A rectangle that has an area equal to that of the triangle has a width of 3 inches. Find the perimeter of the rectangle in inches.

A 30 **B** 24 **C** 22 **D** 16 **E** 11

Extra Practice See p. A55.

HISTORY of MATHEMATICS

CALCULUS

Calculus is fundamental to solving problems in the sciences and engineering. Two basic tools of calculus are differentiation and integration. Some of the basic ideas of calculus began to develop over 2000 years ago, but a usable form was not developed until the seventeenth century.

Tahani R. Amer

Early Evidence Several ideas basic to the development of calculus are the concepts of limit, infinite processes, and approximation. The Egyptians and Babylonians solved problems, such as finding the areas of circles and the volumes of pyramids, by methods resembling calculus. In about 450 B.C., **Zeno** of Elea posed problems, often called Zeno's Paradoxes, dealing with infinity. In trying to deal with these paradoxes, **Eudoxus** (about 370 B.C.), a Greek, proposed his "method of exhaustion," which is based on the idea of infinite processes. An example of this method is to show that the difference in area between a circle and an inscribed polygon can be made smaller and smaller by increasing the number of sides of the polygon.

The Renaissance Mathematicians and scientists, such as **Johann Kepler** (1571–1630), **Pierre Fermat** (1601–1665), **Gilles Roberval** (1602–1675), and **Bonaventura Cavalieri** (1598–1647), used the concept of summing an infinite number of strips to find the area under a curve. Cavalieri called this the "method of indivisibles." The use of coordinates and the development of analytic geometry by Fermat and **Renè Descartes** (1596–1650) aided in the further development of calculus.

Modern Era Most historians name **Gottfried Leibniz** (1646–1716) and **Isaac Newton** (1642–1727) as coinventors of calculus. They worked independently at approximately the same time on ideas which evolved into what is known as calculus today.

In the argument over which mathematician developed calculus first, it seems that Newton had the ideas first, but did not publish them until after Leibniz made his ideas public. However, the notation used by Leibniz was more understandable than that of Newton, and much of it is still in use.

Today aerospace engineers like **Tahani R. Amer** use calculus in many aspects of their jobs. In her job at the NASA Langley Research Center, she uses calculus for characterizing pressure measurements taken during wind tunnel tests of experimental aircraft and for working with optical measurements.

ACTIVITIES

1. Demonstrate the method of exhaustion. Draw three circles of equal radii. In the first circle, inscribe a triangle, in the second a square, and in the third a pentagon. Find the difference between the area of each circle and its inscribed polygon.

2. Fermat discovered a simple method for finding the maximum and minimum points of polynomial curves. Consider the curve $y = 2x^3 - 5x^2 + 4x - 7$. If another point has abscissa $x + E$, then the ordinate is $2(x + E)^3 - 5(x + E)^2 + 4(x + E) - 7$. He set this expression equal to the original function and arrived at the equation $(6x^2 - 10x + 4)E + (6x - 5)E^2 + 2E^3 = 0$. Finish Fermat's method. Divide each term by E. Then let E be 0. What is the relationship between the roots of the resulting equation and the derivative of $2x^3 - 5x^2 + 4x - 7$?

3. interNET CONNECTION Find out more about persons referenced in this article and others who contributed to the history of calculus. Visit **www.amc.glencoe.com**

15-4

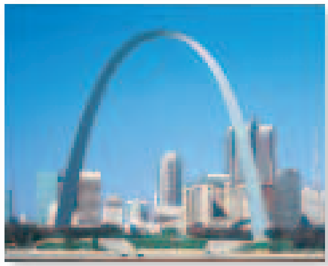
The Fundamental Theorem of Calculus

OBJECTIVES

- Use the Fundamental Theorem of Calculus to evaluate definite integrals of polynomial functions.
- Find indefinite integrals of polynomial functions.

Real World Application

CONSTRUCTION Two construction contractors have been hired to clean the Gateway Arch in St. Louis. The Arch is very close to a parabola in shape, 630 feet high and 630 feet across at the bottom. Using the point on the ground directly below the apex of the Arch as the origin, the equation of the Arch is approximately $y = 630 - \frac{x^2}{157.5}$. One contractor's first idea for approaching the project is to build scaffolding in the entire space under the Arch, so that the cleaning crew can easily climb up and down to any point on the Arch. The other contractor thinks there is too much space under the Arch to make the scaffolding practical. To settle the matter, the contractors want to find out how much area there is under the Arch. *This problem will be solved in Example 4.*



interNET CONNECTION

Research
For more information about the dimensions and shape of the Gateway Arch, visit: **www.amc.glencoe.com**

You have probably found the evaluation of definite integrals with limits to be a tedious process. Fortunately, there is an easier method. Consider, for example, the problem of finding the change in position of a moving object between times $t = a$ and $t = b$. In Lesson 15-3, we solved such a problem by evaluating $\int_a^b f(t)\,dt$, where $f(t)$ is the velocity of the object. Another approach would be to find the position function, which is an antiderivative of $f(t)$, for the object. Substituting a and b into the position function would give the locations of the object at those times. We could subtract those locations to find the displacement of the object. In other words, if $F(t)$ is the position function for the object, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

The above relationship is actually true for *any* continuous function $f(x)$. This connection between definite integrals and antiderivatives is so important that it is called the **Fundamental Theorem of Calculus.**

Fundamental Theorem of Calculus	If $F(x)$ is the antiderivative of the continuous function $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

The Fundamental Theorem of Calculus provides a way to evaluate the definite integral $\int_a^b f(x)\,dx$ if an antiderivative $F(x)$ can be found. A vertical line on the right side is used to abbreviate $F(b) - F(a)$. Thus, the principal statement of the theorem may be written as follows.

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1 **Evaluate $\int_2^4 x^3\,dx$.**

The antiderivative of $f(x) = x^3$ is $F(x) = \frac{1}{4}x^4 + C$.

$\int_2^4 x^3\,dx = \frac{1}{4}x^4 + C \Big|_2^4$ *Fundamental Theorem of Calculus*

$= \left(\frac{1}{4} \cdot 4^4 + C\right) - \left(\frac{1}{4} \cdot 2^4 + C\right)$ *Let $x = 4$ and 2 and subtract.*

$= 64 - 4$ or 60

Notice how much easier this example was than Example 2 of Lesson 15-3. Also notice that C was eliminated during the calculation. This always happens when you use the Fundamental Theorem to evaluate a definite integral. So in this situation you can neglect the constant term when writing the antiderivative.

Due to the connection between definite integrals and antiderivatives, the antiderivative of $f(x)$ is often denoted by $\int f(x)\,dx$. $\int f(x)\,dx$ is called the **indefinite integral** of $f(x)$.

It is helpful to rewrite the antiderivative rules in terms of indefinite integrals.

Antiderivative Rules

Power Rule:	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, where n is a rational number and $n \neq -1$.
Constant Multiple of a Power Rule:	$\int kx^n\,dx = k \cdot \frac{1}{n+1}x^{n+1} + C$, where k is a constant, n is a rational number, and $n \neq -1$.
Sum and Difference Rule:	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example 2 **Evaluate each indefinite integral.**

a. $\int 5x^2\,dx$

$\int 5x^2\,dx = 5 \cdot \frac{1}{3}x^3 + C$ *Constant Multiple of a Power Rule*

$= \frac{5}{3}x^3 + C$ *Simplify.*

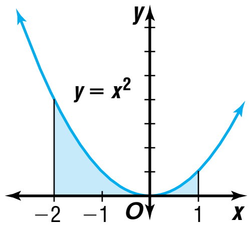
b. $\int (4x^5 + 7x^2 - 4x)\,dx$

$\int (4x^5 + 7x^2 - 4x)\,dx = 4 \cdot \frac{1}{6}x^6 + 7 \cdot \frac{1}{3}x^3 - 4 \cdot \frac{1}{2}x^2 + C$ *Remember $x = x^1$.*

$= \frac{2}{3}x^6 + \frac{7}{3}x^3 - 2x^2 + C$ *Simplify.*

Examples

3 **Find the area of the shaded region.**



The area is given by $\int_{-2}^{1} x^2\,dx$.

The antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3 + C$.

$$\int_{-2}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-2}^{1} \qquad \textit{+ C is not needed with a definite integral.}$$

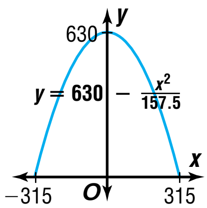
$$= \frac{1}{3}(1)^3 - \frac{1}{3}(-2)^3 \qquad \textit{Let } x = 1 \textit{ and } -2 \textit{ and subtract.}$$

$$= 3$$

The area of the region is 3 square units.

4 **CONSTRUCTION** **Refer to the application at the beginning of the lesson. What is the area under the Gateway Arch?**

Real World Application



The area is given by $\int_{-315}^{315}\left(630 - \frac{x^2}{157.5}\right)dx$.

315 and −315 are the x-intercepts of the parabola that models the Arch.

$$\int_{-315}^{315}\left(630 - \frac{x^2}{157.5}\right)dx = \int_{-315}^{315}\left(630 - \frac{1}{157.5}x^2\right)dx \qquad \textit{Rewrite the function.}$$

$$= 630x - \frac{1}{157.5}\cdot\frac{1}{3}x^3\Big|_{-315}^{315} \qquad \textit{Antiderivative; + C not needed.}$$

$$= \left(630\cdot 315 - \frac{1}{472.5}(315)^3\right) \qquad \textit{Let } x = 315 \textit{ and } -315 \textit{ and subtract.}$$

$$- \left(630\cdot(-315) - \frac{1}{472.5}(-315)^3\right)$$

$$= 132{,}300 - (-132{,}300) \text{ or } 264{,}600$$

The area under the Arch is 264,600 square feet.

Check for Understanding

Communicating Mathematics

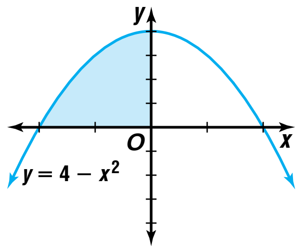
Read and study the lesson to answer each question.

1. **Explain** the difference between $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$.
2. **Find a counterexample** to the statement $\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$ *for all a and b and all functions f(x) and g(x).*
3. **Explain** why the "+ C" is not needed in the antiderivative when evaluating a definite integral.
4. **You Decide** Cole says that when evaluating a definite integral, the order in which you substitute a and b into the antiderivative and subtract does not matter. Rose says it does matter. Who is correct? Explain.

Guided Practice **Evaluate each indefinite integral.**

5. $\int (2x^2 - 4x + 3)\, dx$ **6.** $\int (x^3 + 3x + 1)\, dx$

7. Find the area of the shaded region in the graph at the right.



Find the area between each curve and the *x*-axis for the given interval.

8. $y = x^4$ from $x = 0$ to $x = 2$ **9.** $y = x^2 + 4x + 4$ from $x = -1$ to $x = 1$

Evaluate each definite integral.

10. $\int_1^3 2x^3\, dx$ **11.** $\int_1^4 (x^2 - x + 6)\, dx$

12. $\int_0^2 (-2x^2 + 3x + 2)\, dx$ **13.** $\int_2^4 (x^3 + x - 6)\, dx$

14. Physics The work, in joules (J), required to stretch a certain spring a distance of ℓ meters beyond its natural length is given by $W = \int_0^{\ell} 500x\, dx$. How much work is required to stretch the spring 10 centimeters beyond its natural length?

EXERCISES

Practice **Evaluate each indefinite integral.**

15. $\int x^5\, dx$ **16.** $\int 6x^7\, dx$

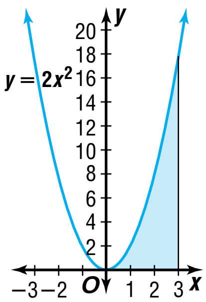
17. $\int (x^2 - 2x + 4)\, dx$ **18.** $\int (-3x^2 - x + 6)\, dx$

19. $\int (x^4 + 2x^2 - 3)\, dx$ **20.** $\int (4x^5 - 6x^3 + 7x^2 - 8)\, dx$

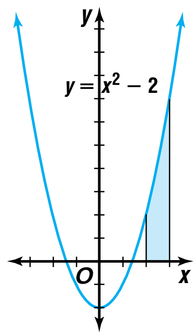
21. Find the antiderivative of $x^2 - 6x + 3$.

Find the area of the shaded region in each graph.

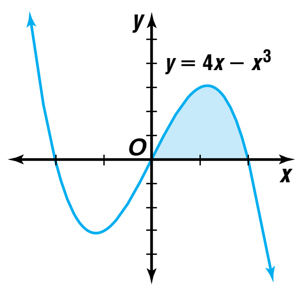
22.



23.



24.



Find the area between each curve and the *x*-axis for the given interval.

25. $y = x^3$ from $x = 0$ to $x = 4$

26. $y = 3x^6$ from $x = -1$ to $x = 1$

27. $y = x^2 - 2x$ from $x = -2$ to $x = 0$

28. $y = -x^2 + 2x + 3$ from $x = 1$ to $x = 3$

29. $y = x^3 + x$ from $x = 0$ to $x = 1$

30. $y = x^3 + 8x + 10$ from $x = -1$ to $x = 3$

Evaluate each definite integral.

31. $\int_0^7 6x^2\,dx$

32. $\int_2^4 3x^4\,dx$

33. $\int_{-1}^3 (x + 4)\,dx$

34. $\int_1^5 (3x^2 - 2x + 1)\,dx$

35. $\int_1^3 (x^3 - x^2)\,dx$

36. $\int_0^1 (x^4 + 2x^2 + 1)\,dx$

37. $\int_{-1}^0 (x^4 - x^3)\,dx$

38. $\int_0^2 (x^3 + x + 1)\,dx$

39. $\int_{-2}^5 (x^2 - 3x + 8)\,dx$

40. $\int_1^3 (x + 3)(x - 1)\,dx$

41. $\int_2^3 (x - 1)^3\,dx$

42. $\int_0^1 \frac{x^2 - x - 2}{x - 2}\,dx$

43. Find the integral of $x(4x^2 + 1)$ from 0 to 2.

44. What is the integral of $(x + 1)(3x + 2)$ from -1 to 1?

45. The integral $\int_0^{n+0.5} x^k\,dx$ gives a fairly close, quick estimate of the sum of the series $\sum_{i=1}^{n} i^k$. Use the integral to estimate each sum and then find the actual sum.

a. $\sum_{i=1}^{20} i^3$

b. $\sum_{i=1}^{100} i^2$

Applications and Problem Solving

46. Physics The work (in joules) required to pump all of the water out of a 10 meter by 5 meter by 2 meter swimming pool is given by $\int_0^2 490{,}000x\,dx$. Evaluate this integral to find the required work.

47. Critical Thinking

a. Suppose $f(x)$ is a function whose graph is *below* the *x*-axis for $a \le x \le b$. What can you say about the values of $f(x)$, $\sum_{i=1}^{n} f(x_i)\Delta x$, and $\int_a^b f(x)\,dx$?

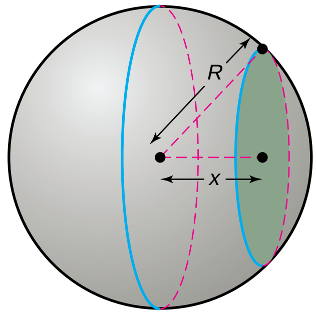
b. Evaluate $\int_0^2 (x^2 - 5)\,dx$.

c. What is the area between the graph of $y = x^2 - 5$ and the *x*-axis from $x = 0$ to $x = 2$?

48. Critical Thinking Find the value of $\int_2^5 (3x - 6)\,dx$ *without* using limits or the Fundamental Theorem of Calculus.

49. **Stock Market** The *average value* of a function $f(x)$ over the interval $a \leq x \leq b$ is defined to be $\frac{1}{b-a}\int_a^b f(x)\,dx$. A stock market analyst has determined that the price of the stock of the Acme Corporation over the year 2001 can be modeled by the function $f(x) = 75 + 8x - \frac{1}{2}x^2$, where x is the time, in months, since the beginning of 2001, and $f(x)$ is in dollars.
   a. Sketch a graph of $f(x)$ from $x = 0$ to $x = 12$.
   b. Find the average value of the Acme Corporation stock over the first half of 2001.
   c. Find the average value of the stock over the second half of 2001.

50. **Geometry** The volume of a sphere of radius R can be found by slicing the sphere vertically and then integrating the areas of the resulting circular cross sections. (The cross section in the figure is a circle of radius $\sqrt{R^2 - x^2}$.) This process results in the integral $\int_{-R}^{R} (\pi R^2 - \pi x^2)\,dx$. Evaluate this integral to obtain the expression for the volume of a sphere of radius R.



51. **Space Exploration** The weight of an object that is at a distance x from the center of Earth can be written as kx^{-2}, where k is a constant that depends on the mass of the object. The energy required to move the object from $x = a$ to $x = b$ is the integral of its weight, that is, $\int_a^b kx^{-2}\,dx$. Suppose a Lunar Surveying Module (LSM), designed to analyze the surface of the moon, weighs 1000 newtons on the surface of Earth.
   a. Find k for the LSM. Use 6.4×10^6 meters for the radius of Earth.
   b. Find the energy required to lift the LSM from Earth's surface to the moon, 3.8×10^8 meters from the center of Earth.

Mixed Review

52. Use a limit to evaluate $\int_0^2 \frac{1}{2}x^2\,dx$. *(Lesson 15-3)*

53. Find the derivative of $f(x) = 2x^6 - 3x^2 + 2$. *(Lesson 15-2)*

54. **Education** The scores of a national achievement test are normally distributed with a mean of 500 and a standard deviation of 100. What percent of those who took the test had a score more than 100 points above or below the mean? *(Lesson 14-4)*

55. Fifty tickets, numbered consecutively from 1 to 50 are placed in a box. Four tickets are drawn without replacement. What is the probability that four odd numbers are drawn? *(Lesson 13-4)*

56. **Banking** Find the amount accumulated if \$600 is invested at 6% for 15 years and interest is compounded continuously. *(Lesson 11-3)*

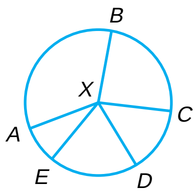
57. Write an equation of the parabola with vertex at (6, −1) and focus at (3, −1). *(Lesson 10-5)*

58. Find $2\sqrt{2}\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right) \div \sqrt{2}\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$. Then express the result in rectangular form. *(Lesson 9-7)*

Extra Practice See p. A55.

59. Find the initial vertical velocity of a stone thrown with an initial velocity of 45 feet per second at an angle of 52° with the horizontal. *(Lesson 8-7)*

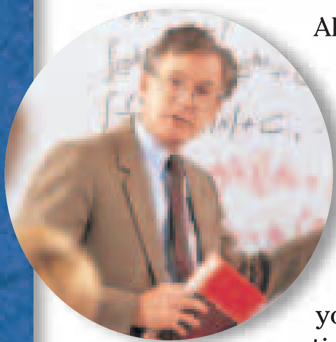
60. SAT Practice In the circle with center X, $\widehat{AE}$ is the shortest of the five unequal arcs. Which statement best describes the measure of angle AXE?



A less than 72°
B equal to 72°
C greater than 72°, but less than 90°
D greater than 90°, but less than 180°
E greater than 180°

CAREER CHOICES

Mathematician



Algebra, geometry, trigonometry, statistics, calculus—if you enjoy studying these subjects, then a career in mathematics may be for you. As a mathematician, you would have several options for employment.

First, a theoretical mathematician develops new principles and discovers new relationships, which may be purely abstract in nature. Applied mathematicians use new ideas generated by theoretical mathematicians to solve problems in many fields, including science, engineering and business. Mathematicians may work in related fields such as computer science, engineering, and business. As a mathematician, you can become an elementary or secondary teacher if you obtain a teaching certificate. An advanced degree is required to teach at the college level.

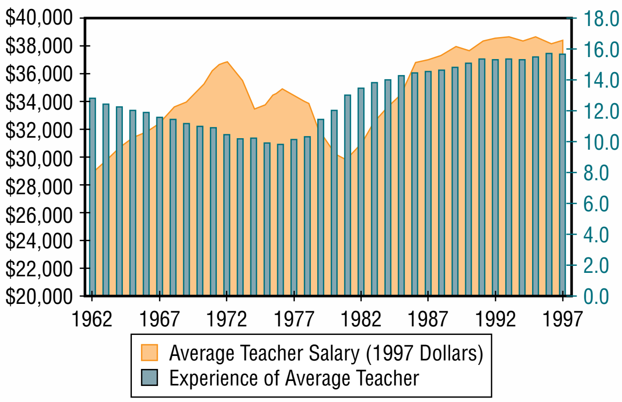
Career Overview

Degree Preferred:
bachelor's degree in mathematics

Related Courses:
mathematics, science, computer science

Outlook:
increased demand for teachers and math-related occupations through the year 2006

The Average Teacher Salary Compared to the Average Experience Level of Teachers



Source: American Federation of Teachers

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VOCABULARY

antiderivative (p. 955)
definite integral (p. 962)
derivative (p. 951)
differentiation (p. 952)
Fundamental Theorem of Calculus (p. 970)
indefinite integral (p. 971)
integral (p. 961)
integration (p. 962)
limit (p. 941)
rate of change (p. 956)
secant line (p. 951)
slope of a curve (p. 949)
tangent line (p. 951)

UNDERSTANDING AND USING THE VOCABULARY

State whether each sentence is *true* or *false.* If false, replace the underlined word(s) to make a true statement.

1. $f(a)$ and $\lim_{x \to a} f(x)$ are always the same.
2. The process of finding the area under a curve is called integration.
3. The inverse of finding the derivative of a function is finding the definite integral.
4. The Fundamental Theorem of Calculus can be used to evaluate a definite integral.
5. A line that intersects a graph in two points is called a tangent line.
6. A line that passes through a point on a curve and has a slope equal to the slope of the curve at that point is called a secant line.
7. The conjugate of a function $f(x)$ is another function $f'(x)$ that gives the slope of the tangent line to $y = f(x)$ at any point.
8. If you look at one particular point on the graph of a curve, there is a certain steepness, called the slope, at that point.
9. The derivative of a function can also be called the domain of the function because it measures how fast the function changes.
10. The process of finding a limit is called differentiation.

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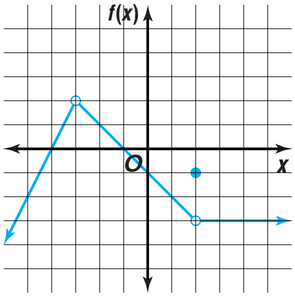
SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 15-1 Calculate limits of polynomial and rational functions algebraically.

- Consider the graph of the function $y = f(x)$ shown below. Find $f(-3)$ and $\lim_{x \to -3} f(x)$.



There is no point on the graph with an x-coordinate of -3, so $f(-3)$ is undefined.

Look at points on the graph whose x-coordinates are close to, but not equal to, -3. The closer x is to -3, the closer y is to 2. So, $\lim_{x \to -3} f(x) = 2$.

Evaluate each limit.

a. $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 + 3x}$

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 + 3x} = \lim_{x \to 2} \frac{(x + 3)(x - 2)}{x(x + 3)}$$
$$= \lim_{x \to 2} \frac{x - 2}{x}$$
$$= \frac{2 - 2}{2}$$
$$= 0$$

b. $\lim_{x \to 0} \frac{x \cos x}{x}$

$$\lim_{x \to 0} \frac{x \cos x}{x} = \lim_{x \to 0} \cos x$$
$$= \cos 0$$
$$= 1$$

REVIEW EXERCISES

11. Refer to the graph of $y = f(x)$ at the left. Find $f(2)$ and $\lim_{x \to 2} f(x)$.

Evaluate each limit.

12. $\lim_{x \to -2} (x^3 - x^2 - 5x + 6)$

13. $\lim_{x \to 0} (2x - \cos x)$

14. $\lim_{x \to 1} \frac{x^2 - 36}{x + 6}$

15. $\lim_{x \to 0} \frac{5x^2}{2x}$

16. $\lim_{x \to 4} \frac{x^2 + 2x}{x^2 - 3x - 10}$

17. $\lim_{x \to 0} (x + \sin x)$

18. $\lim_{x \to 0} \frac{x^2 + x \cos x}{2x}$

19. $\lim_{x \to 2} \frac{x^3 + 2x^2 - 4x - 8}{x^2 - 4}$

20. $\lim_{x \to 0} \frac{(x - 3)^2 - 9}{2x}$

21. $\lim_{x \to 5} \frac{x^2 - 9x + 20}{x^2 - 5x}$

CONTENTS

OBJECTIVES AND EXAMPLES

Lesson 15-2 Find derivatives of polynomial functions.

Find the derivative of each function.

a. $f(x) = 3x^4 + 2x^3 - 7x - 5$

$$f'(x) = 3 \cdot 4x^3 + 2 \cdot 3x^2 - 7 \cdot 1 - 0$$
$$= 12x^3 + 6x^2 - 7$$

b. $f(x) = 2x^3(x^2 + 1)$

First, multiply to write the function as a polynomial.

$$f(x) = 2x^3(x^2 + 1)$$
$$= 2x^5 + 2x^3$$

Then find the derivative.

$$f'(x) = 10x^4 + 6x^2$$

REVIEW EXERCISES

Use the definition of derivative to find the derivative of each function.

22. $f(x) = 2x + 1$

23. $f(x) = 4x^2 + 3x - 5$

24. $f(x) = x^3 - 3x$

Use the derivative rules to find the derivative of each function.

25. $f(x) = 2x^6$

26. $f(x) = -3x + 7$

27. $f(x) = 3x^2 - 5x$

28. $f(x) = \frac{1}{4}x^2 - x + 4$

29. $f(x) = \frac{1}{2}x^4 - 2x^3 + \frac{1}{3}x - 4$

30. $f(x) = (x + 3)(x + 4)$

31. $f(x) = 5x^3(x^4 - 3x^2)$

32. $f(x) = (x - 2)^3$

Lesson 15-2 Find antiderivatives of polynomial functions.

Find the antiderivative of each function.

a. $f(x) = 5x^2$

$$F(x) = 5 \cdot \frac{1}{2 + 1}x^{2+1} + C$$
$$= \frac{5}{3}x^3 + C$$

b. $f(x) = -2x^3 + 6x^2 - 5x + 4$

$$F(x) = -2 \cdot \frac{1}{4}x^4 + 6 \cdot \frac{1}{3}x^3 - 5 \cdot \frac{1}{2}x^2 + 4 \cdot x + C$$
$$= -\frac{1}{2}x^4 + 2x^3 - \frac{5}{2}x^2 + 4x + C$$

Find the antiderivative of each function.

33. $f(x) = 8x$

34. $f(x) = 3x^2 + 2$

35. $f(x) = -\frac{1}{2}x^3 + 2x^2 - 3x - 2$

36. $f(x) = x^4 - 5x^3 + 2x - 6$

37. $f(x) = (x - 4)(x + 2)$

38. $f(x) = \frac{x^2 - x}{x}$

OBJECTIVES AND EXAMPLES

Lesson 15-3 Find areas under graphs of polynomial functions.

Use limits to find the area of the region between the graph of $y = 3x^2$ and the x-axis from $x = 0$ to $x = 1$.

$$\int_0^1 3x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} 3(x_i)^2 \Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} 3\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3i^2}{n^3}$$
$$= \lim_{n\to\infty} \frac{3}{n^3}(1^2 + 2^2 + \cdots + n^2)$$
$$= \lim_{n\to\infty} \frac{3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$
$$= \lim_{n\to\infty} \frac{1}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$
$$= 1 + 0 + 0 \text{ or } 1 \text{ unit}^2$$

REVIEW EXERCISES

Use limits to find the area between each curve and the x-axis for the given interval.

39. $y = 2x$ from $x = 0$ to $x = 2$

40. $y = x^3$ from $x = 0$ to $x = 1$

41. $y = x^2$ from $x = 3$ to $x = 4$

42. $y = 6x^2$ from $x = 1$ to $x = 2$

Lesson 15-4 Use the Fundamental Theorem of Calculus to evaluate definite integrals of polynomial functions.

Evaluate $\int_4^7 (x^2 - 3)\,dx$.

$$\int_4^7 (x^2 - 3)\,dx$$
$$= \frac{1}{3}x^3 - 3x\Big|_4^7$$
$$= \left(\frac{1}{3}\cdot 7^3 - 3\cdot 7\right) - \left(\frac{1}{3}\cdot 4^3 - 3\cdot 4\right)$$
$$= 84$$

Evaluate each definite integral.

43. $\int_2^4 6x\,dx$

44. $\int_{-3}^{2} 3x^2\,dx$

45. $\int_{-2}^{2} (3x^2 - x + 3)\,dx$

46. $\int_0^4 (x + 2)(2x + 3)\,dx$

Lesson 15-4 Find indefinite integrals of polynomial functions.

Evaluate $\int (6x^2 - 4x)\,dx$.

$$\int (6x^2 - 4x)\,dx = 6\cdot\frac{1}{3}x^3 - 4\cdot\frac{1}{2}x^2 + C$$
$$= 2x^3 - 2x^2 + C$$

Evaluate each indefinite integral.

47. $\int 6x^4\,dx$

48. $\int (-3x^2 + 2x)\,dx$

49. $\int (x^2 + 5x - 2)\,dx$

50. $\int (3x^5 + 4x^4 - 7x)\,dx$

APPLICATIONS AND PROBLEM SOLVING

51. Physics The kinetic energy of an object with mass m is given by the formula $k(t) = \frac{1}{2} m \cdot v(t)^2$, where $v(t)$ is the velocity of the object at time t. Suppose $v(t) = \frac{50}{1 + t^2}$ for all $t \geq 0$. What does the kinetic energy of the object approach as time approaches 100? *(Lesson 15-1)*

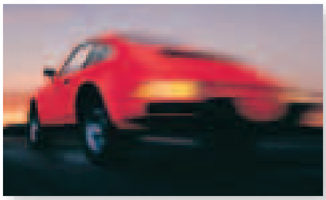
52. Business The controller for an electronics company has used the production figures for the last few months to determine that the function $c(x) = -9x^5 + 135x^3 + 10{,}000$ approximates the cost of producing x thousands of one of their products. Find the marginal cost if they are now producing 2600 units. *(Lesson 15-2)*

53. Motion An advertisement for a sports car claims that the car can accelerate from 0 to 60 miles per hour in 5 seconds. *(Lesson 15-2)*

a. Find the acceleration of the sports car in feet per second squared, assuming that it is constant.

b. Write an equation for the velocity of the sports car at t seconds.

c. Write an equation for the distance traveled in t seconds.



ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. The limit of a continuous function $f(x)$ as x approaches 1 is 5. Give an example of a function for which this is true. Show why the limit of your function as x approaches 1 is 5.
2. The area of the region between the graph of the function $g(x)$ and the x-axis from $x = 0$ to $x = 1$ is 4. Give an example of a function for which this is true. Show that the area of the region between the graph of your function and the x-axis from $x = 0$ to 1 is in fact 4.

PORTFOLIO

Explain the difference between a definite integral and an indefinite integral. Give an example of each.

Now that you have completed your work in this book, review your portfolio entries for each chapter. Make any necessary changes or corrections. Add a table of contents to your portfolio at this time.

Unit 5 interNET Project

DISEASES

- Use the Internet to find the number of cases reported or the number of deaths for one particular disease for a period of at least 10 years. Some possible diseases you might choose to research are measles, tuberculosis, or AIDS. Make a table or spreadsheet of the data.
- Use computer software or a graphing calculator to find at least two polynomial functions that model the data. Find the derivative for each of your function models. What does the derivative represent?
- Use each model to predict the cases or deaths from the disease in the year 2010. Write a one-page paper comparing the models. Discuss which model you think best fits the data. Include any limitations of the model.

Additional Assessment See p. A70 for Chapter 15 Practice Test.

Chapter 15 SAT & ACT Preparation

THE PRINCETON REVIEW

Special symbols can appear in the question or in the answer choices or both.

Read the explanation thoroughly and work carefully.

Special Function and Counting Problems

The SAT includes function problems that use special symbols like ⊕ or # or ⊗. (The ACT does not contain this type of problem.)

Here's a simple example: If $x \# y = x + 2y$, then what is $2 \# 5$? To find $2 \# 5$, replace x with 2 and y with 5. Thus, $2 \# 5 = 2 + 2(5) = 12$. The SAT may also include problems that involve counting regions, surfaces, or intersections. The questions usually ask for the maximum or minimum number.

SAT EXAMPLE

1. Let $\boxed{x}$ be defined for all positive integers x as the product of the distinct prime factors of x. What is the value of

$$\frac{\boxed{6}}{\boxed{81}}?$$

HINT The SAT often combines two mathematical concepts in one problem. For example, this problem combines a special function and prime factors.

Solution Carefully read the definition of $\boxed{x}$. Recall the meaning of "distinct prime factors." Write the prime factorization of each number, identify which prime factors are distinct, and then find the product.

Start with the first number, 6. $6 = 2 \times 3$. Both 2 and 3 are distinct prime factors. The product of the distinct prime factors is 6.

Do the same with 81. $81 = 3 \times 3 \times 3 \times 3$. There is just one distinct prime factor, 3. So the product of the distinct prime factors is also 3.

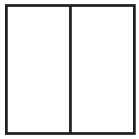
Finally, substitute the values for $\boxed{6}$ and $\boxed{81}$ into the fraction.

$\frac{\boxed{6}}{\boxed{81}} = \frac{6}{3} = 2$. The answer is 2.

Grid-in this answer on your answer sheet.

SAT EXAMPLE

2. The figure below is a square separated into two non-overlapping regions. What is the greatest number of non-overlapping regions that can be made by drawing any two additional straight lines?

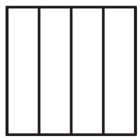


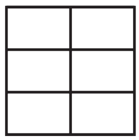
A 4 **B** 5 **C** 6 **D** 7 **E** 8

HINT Watch out for "obvious" answers on difficult problems (those numbered 18 or higher). They are usually wrong answers.

Solution Draw right on your test booklet.

The most obvious ways to draw two more lines are shown at the right.

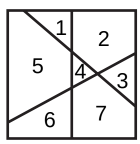




The first figure has 4 regions; the second figure has 6 regions. So you can immediately eliminate answer choices A and B.

For the maximum number of regions, it is likely that the lines will *not* be parallel, as they are in the figures above.

Draw the two lines with the fewest possible criteria: not horizontal, not vertical, not parallel, and not perpendicular.



There are 7 regions. The answer is choice **D**.

SAT AND ACT PRACTICE

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. If $x \otimes y = \frac{1}{x - y}$, what is the value of $\frac{1}{2} \otimes \frac{1}{3}$?

A 6 **B** $\frac{6}{5}$ **C** $\frac{1}{6}$

D -1 **E** -6

2. If one side of a triangle is twice as long as a second side of length x, then the perimeter of the triangle can be:

A $2x$ **B** $3x$ **C** $4x$

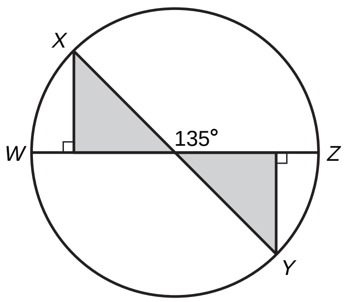
D $5x$ **E** $6x$

3. If 3 parallel lines are cut by 3 nonparallel lines, what is the maximum number of intersections possible?

A 9 **B** 10 **C** 11

D 12 **E** 13

4. In the figure below, if segment $\overline{WZ}$ and segment $\overline{XY}$ are diameters with lengths of 12, what is the area of the shaded region?



A 9 **B** 18 **C** 36

D 54 **E** 108

5. Which of the following represents the values of x that are solutions of the inequality $x^2 < x + 6$?

A $x > -2$

B $x < 3$

C $-2 < x < 3$

D $-3 < x < 2$

E $x < -2 \quad \cup \quad x > 3$

6. $\boxed{x} = \frac{1}{2}x$ if x is composite.

$\boxed{x} = 3x$ if x is prime.

What is the value of $\boxed{5} + \boxed{16}$?

A 21 **B** 23 **C** 31

D 46 **E** 69

7. What is the average of all the integers from 1 to 20 inclusive?

A 9.5 **B** 10 **C** 10.5

D 20 **E** 21

8. All faces of a cube with a 4-meter edge are painted blue. If the cube is then cut into cubes with 1-meter edges, how many of the 1-meter cubes have blue paint on exactly one face?

A 24 **B** 36 **C** 48

D 60 **E** 72

9. For all numbers n, let $\{n\}$ be defined as $n^2 - 1$. What is the value of $\{\{x\}\}$?

A $x^2 - 1$

B $x^4 - 1$

C $x^4 - 2x^2 - 1$

D $x^4 - 2x^2$

E x^4

10. Grid-In Let $\boxed{x}$ be defined for all positive integers x as the product of the distinct prime factors of x. What is the value of

$$\frac{\boxed{20}}{\boxed{16}}?$$

interNET CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**

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