Sigma Notation and the nth Term

OBJECTIVE
* Use sigma notation.

Real World Application

MANUFACTURING
Manufacturers are required by the Environmental Protection Agency to meet certain emission standards. If these standards are not met by a preassigned date, the manufacturer is fined. To encourage swift compliance, the fine increases a specified amount each day until the manufacturer is able to pass inspection. Suppose a manufacturing plant is charged $2000 for not meeting its January 1st deadline. The next day it is charged $2500, the next day $3000, and so on, until it passes inspection on January 21st. What is the total amount of fines owed by the manufacturing plant? *This problem will be solved in Example 2.*

In mathematics, the uppercase Greek letter sigma, Σ, is often used to indicate a sum or series. A series like the one indicated above may be written using **sigma notation.**

Other variables besides n may be used for the index of summation.

$$\text{maximum value of } n \longrightarrow \overset{k}{\underset{n=1}{\sum}} a_n \longleftarrow \text{expression for general term}$$
$$\text{starting value of } n \longrightarrow$$

The variable n used with the sigma notation is called the **index of summation.**

Sigma Notation of a Series	For any sequence a_1, a_2, a_3, ..., the sum of the first k terms may be written $\overset{k}{\underset{n=1}{\sum}} a_n$, which is read "the summation from $n = 1$ to k of a_n." Thus, $$\overset{k}{\underset{n=1}{\sum}} a_n = a_1 + a_2 + a_3 + \cdots + a_k, \text{ where } k \text{ is an integer value.}$$

Example

Real World Application

1 Write each expression in expanded form and then find the sum.

a. $\displaystyle\sum_{n=1}^{4} (n^2 - 3)$

First, write the expression in expanded form.

$$\overset{4}{\underset{n=1}{\sum}} (n^2 - 3) = \overset{n=1}{(1^2 - 3)} + \overset{n=2}{(2^2 - 3)} + \overset{n=3}{(3^2 - 3)} + \overset{n=4}{(4^2 - 3)}$$

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Now find the sum.

Method 1 Simplify the expanded form.
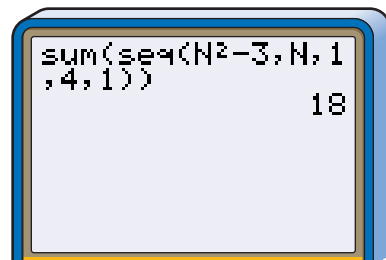$$(1^2 - 3) + (2^2 - 3) + (3^2 - 3) + (4^2 - 3) = -2 + 1 + 6 + 13 \text{ or } 18$$

Method 2 Use a graphing calculator.

You can combine **sum(** with **seq(** to obtain the sum of a finite sequence. In the **LIST** menu, select the **MATH** option to locate the **sum(** command. The **seq(** command can be found in the **OPS** option of the **LIST** menu.

$$\text{sum(seq(} \underbrace{N^2 - 3}_{\text{expression}} , \underset{\substack{\uparrow \\ \text{index of} \\ \text{summation}}}{N} , \overset{\substack{\text{starting} \\ \text{value} \\ \downarrow}}{1} , \underset{\substack{\uparrow \\ \text{maximum} \\ \text{value}}}{4} , \overset{\substack{\text{step} \\ \text{size} \\ \downarrow}}{1}))$$

```
sum(seq(N²-3,N,1
,4,1))
                18
```

b. $\displaystyle\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1}$

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1} = \overset{n=1}{5\left(-\frac{2}{7}\right)^{1-1}} + \overset{n=2}{5\left(-\frac{2}{7}\right)^{2-1}} + \overset{n=3}{5\left(-\frac{2}{7}\right)^{3-1}} + \overset{n=4}{5\left(-\frac{2}{7}\right)^{4-1}} + \cdots$$

$$= \quad 5 \quad + \quad \frac{10}{7} \quad + \quad \frac{20}{49} \quad + \quad \frac{40}{343} \quad + \cdots$$

This is an infinite geometric series. Use the formula $S = \dfrac{a_1}{1-r}$.

$$S = \frac{5}{1 - \left(-\frac{2}{7}\right)} \qquad a_1 = 5, \ r = -\frac{2}{7}$$

$$S = \frac{35}{9}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1} = \frac{35}{9}$.

A series in expanded form can be written using sigma notation if a general formula can be written for the *n*th term of the series.

Example **2** **MANUFACTURING** Refer to the application at the beginning of this lesson.

a. How much is the company fined on the 20th day?

b. What is the total amount in fines owed by the manufacturing plant?

c. Represent this sum using sigma notation.

a. Since this sequence is arithmetic, we can use the formula for the *n*th term of an arithmetic sequence to find the amount of the fine charged on the 20th day.

$$a_n = a_1 + (n-1)d$$
$$a_{20} = 2000 + (20-1)500 \quad a_1 = 2000, \ n = 20, \text{ and } d = 500$$
$$= 11{,}500$$

The fine on the 20th day will be \$11,500.

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b. To determine the total amount owed in fines, we can use the formula for the sum of an arithmetic series. The plant will not be charged for the day it passes inspection, so it is assessed 20 days in fines.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$= \frac{20}{2}(2000 + 11{,}500) \quad \textit{n = 20, } a_1 \textit{ = 2000, and } a_{20} \textit{ = 11,500}$$

$$= 135{,}000$$

The plant must pay a total of $135,000 in fines.

c. To determine the fine for the nth day, we can again use the formula for the nth term of an arithmetic sequence.

$$a_n = a_1 + (n - 1)d$$

$$= 2000 + (n - 1)500 \quad \textit{a}_1 \textit{ = 2000 and d = 500}$$

$$= 2000 + 500n - 500$$

$$= 500n + 1500$$

The fine on the nth day is $500n + 1500.

Since $2000 is the fine on the first day and $12,000 is the fine on the 20th day, the index of summation goes from $n = 1$ to $n = 20$.

Therefore, $2000 + 2500 + 3000 + \cdots + 11{,}500 = \sum_{n=1}^{20} (500n + 1500)$ or

$$\sum_{n=1}^{20} 500(n + 3).$$

When using sigma notation, it is not always necessary that the sum start with the index equal to 1.

Example ③ **Express the series $15 + 24 + 35 + 48 + \cdots + 143$ using sigma notation.**

Notice that each term is 1 less than a perfect square. Thus, the nth term of the series is $n^2 - 1$. Since $4^2 - 1 = 15$ and $12^2 - 1 = 143$, the index of summation goes from $n = 4$ to $n = 12$.

Therefore, $15 + 24 + 35 + 48 + \cdots + 143 = \sum_{n=4}^{12} (n^2 - 1)$.

As you have seen, not all sequences are arithmetic or geometric. Some important sequences are generated by products of consecutive integers. The product $n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$ is called **n factorial** and symbolized $n!$.

n Factorial	The expression $n!$ (n factorial) is defined as follows for n, an integer greater than zero. $$n! = n(n - 1)(n - 2) \cdots 1$$

By definition 0! = 1.

Factorial notation can be used to express the general form of a series.

Example **4** **Express the series** $\dfrac{2}{2} - \dfrac{4}{6} + \dfrac{6}{24} - \dfrac{8}{120} + \dfrac{10}{720}$ **using sigma notation.**

The sequence representing the numerators is 2, 4, 6, 8, 10. This is an arithmetic sequence with a common difference of 2. Thus the nth term can be represented by $2n$.

Because the series has alternating signs, one factor for the general term of the series is $(-1)^{n+1}$. Thus, when n is odd, the terms are positive, and when n is even, the terms are negative.

The sequence representing the $\qquad\qquad$ $2! = 2$
denominators is 2, 6, 24, 120, 720. $\qquad$ $3! = 6$
This sequence is generated by factorials. $\quad$ $4! = 24$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $5! = 120$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $6! = 720$

Therefore, $\dfrac{2}{2} - \dfrac{4}{6} + \dfrac{6}{24} - \dfrac{8}{120} + \dfrac{10}{720} = \displaystyle\sum_{n=1}^{5} \dfrac{(-1)^{n+1}\, 2n}{(n+1)!}.$

You can check this answer by substituting values of n into the general term.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Find a counterexample** to the following statement: "The summation notion used to represent a series is unique."

2. In Example 4 of this lesson, the alternating signs of the series were represented by a factor of $(-1)^{n+1}$.
   a. **Write** a different factor that could have been used in the general form of the nth term of the series.
   b. **Determine** a factor that could be used if the alternating signs of the series began with a negative first term.

3. Consider the series $\displaystyle\sum_{j=2}^{10} (-2j + 1)$.
   a. **Identify** the number of terms in this series.
   b. **Write** a formula that determines the number of terms t in a finite series if the index of summation has a minimum value of a and a maximum value of b.
   c. **Use** the formula in part **b** to identify the number of terms in the series $\displaystyle\sum_{k=-2}^{3} \dfrac{1}{k+3}.$
   d. **Verify** your answer in part **c** by writing the series $\displaystyle\sum_{k=-2}^{3} \dfrac{1}{k+3}$ in expanded form.

Guided Practice

Write each expression in expanded form and then find the sum.

4. $\displaystyle\sum_{n=1}^{6} (n - 3)$ $\qquad$ 5. $\displaystyle\sum_{k=2}^{5} 4k$ $\qquad$ 6. $\displaystyle\sum_{a=0}^{4} \dfrac{1}{2^a}$ $\qquad$ 7. $\displaystyle\sum_{p=0}^{\infty} 5\left(\dfrac{3}{4}\right)^p$

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Express each series using sigma notation.

8. $5 + 10 + 15 + 20 + 25$　　　　　　**9.** $2 + 4 + 10 + 28$

10. $2 - 4 - 10 - 16$　　**11.** $\dfrac{3}{4} + \dfrac{3}{8} + \dfrac{3}{16} + \dfrac{3}{32} + \cdots$　**12.** $-3 + 9 - 27 + \cdots$

13. Aviation　Each October Albuquerque, New Mexico, hosts the Balloon Fiesta. In 1998, 873 hot air balloons participated in the opening day festivities. One of these balloons rose 389 feet after 1 minute. Because the air in the balloon was not reheated, each succeeding minute the balloon rose 63% as far as it did the previous minute.

　a. Use sigma notation to represent the height of the balloon above the ground after one hour. Then calculate the total height of the balloon after one hour to the nearest foot.

　b. What was the maximum height achieved by this balloon?

EXERCISES

Practice

Write each expression in expanded form and then find the sum.

14. $\displaystyle\sum_{n=1}^{4} (2n - 7)$　　　　**15.** $\displaystyle\sum_{a=2}^{5} 5a$　　　　**16.** $\displaystyle\sum_{b=3}^{8} (6 - 4b)$

17. $\displaystyle\sum_{k=2}^{6} (k + k^2)$　　　**18.** $\displaystyle\sum_{n=5}^{8} \dfrac{n}{n - 4}$　　　**19.** $\displaystyle\sum_{j=4}^{8} 2^j$

20. $\displaystyle\sum_{m=0}^{3} 3^{m-1}$　　　**21.** $\displaystyle\sum_{r=1}^{3} \left(\dfrac{1}{2} + 4^r\right)$　　**22.** $\displaystyle\sum_{i=3}^{5} (0.5)^{-i}$

23. $\displaystyle\sum_{k=3}^{7} k!$　　　　**24.** $\displaystyle\sum_{p=0}^{\infty} 4(0.75)^p$　　**25.** $\displaystyle\sum_{n=1}^{\infty} 4\left(\dfrac{2}{5}\right)^n$

26. Write $\displaystyle\sum_{n=2}^{5} n + i^n$ in expanded form. Then find the sum.

Express each series using sigma notation.

27. $6 + 9 + 12 + 15$　　　　　　**28.** $1 + 4 + 16 + \cdots + 256$

29. $8 + 10 + 12 + \cdots + 24$　　　　**30.** $-8 + 4 - 2 + 1$

31. $10 + 50 + 250 + 1250$　　　　**32.** $13 + 9 + 5 + 1$

33. $\dfrac{1}{9} + \dfrac{1}{14} + \dfrac{1}{19} + \cdots + \dfrac{1}{49}$　　**34.** $\dfrac{2}{3} + \dfrac{4}{5} + \dfrac{8}{7} + \dfrac{16}{9} + \cdots$

35. $4 - 9 + 16 - 25 + \cdots$　　　**36.** $5 + 5 + \dfrac{5}{2} + \dfrac{5}{6} + \dfrac{5}{24} + \cdots$

37. $-32 + 16 - 8 + 4 - \cdots$　　　**38.** $2 + \dfrac{6}{2} + \dfrac{24}{3} + \dfrac{120}{4} + \cdots$

39. $\dfrac{1}{5} + \dfrac{2}{7} + \dfrac{3}{11} + \dfrac{4}{19} + \dfrac{5}{35} + \cdots$　　**40.** $\dfrac{3}{9 \cdot 2} + \dfrac{8}{27 \cdot 6} + \dfrac{15}{81 \cdot 24} + \cdots$

41. Express the series $\dfrac{\sqrt{2}}{3} + \dfrac{2}{6} + \dfrac{\sqrt{8}}{18} + \dfrac{4}{72} + \dfrac{\sqrt{32}}{360} + \cdots$ using sigma notation.

Simplify. Assume that n and m are positive integers, $a > b$, and $a > 2$.

42. $\dfrac{(a - 2)!}{a!}$　　　　**43.** $\dfrac{(a + 1)!}{(a - 2)!}$　　　　**44.** $\dfrac{(a + b)!}{(a + b - 1)!}$

Graphing Calculator

45. Use a graphing calculator to find the sum of $\displaystyle\sum_{n=1}^{100} \dfrac{8n^3 - 2n^2 + 5}{n^4}$. Round to the nearest hundredth.

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46. Advertising   A popular shoe manufacturer is planning to market a new style of tennis shoe in a city of 500,000 people. Using a prominent professional athlete as their spokesperson, the company's ad agency hopes to induce 35% of the people to buy the product. The ad agency estimates that these satisfied customers will convince 35% of 35% of 500,000 to buy a pair of shoes, and those will persuade 35% of 35% of 35% of 500,000, and so on.

   a. Model this situation using sigma notation.

   b. Find the total number of people that will buy the product as a result of the advertising campaign.

   c. What percentage of the population is this?

   d. What important assumption does the advertising agency make in proposing the figure found in part **b** to the shoe manufacturer?

47. Critical Thinking   Solve each equation for x.

   a. $\displaystyle\sum_{n=1}^{6} (x - 3n) = -3$
           b. $\displaystyle\sum_{n=0}^{5} n(n - x) = 25$

48. Critical Thinking   Determine whether each equation is *true* or *false*. Explain your answer.

   a. $\displaystyle\sum_{k=3}^{7} 3^k + \sum_{b=7}^{9} 3^b = \sum_{a=3}^{9} 3^a$
           b. $\displaystyle\sum_{n=2}^{8} (2n - 3) = \sum_{m=3}^{9} (2m - 5)$

   c. $\displaystyle 2\sum_{n=3}^{7} n^2 = \sum_{n=3}^{7} 2n^2$
           d. $\displaystyle\sum_{k=1}^{10} (5 + k) = \sum_{p=0}^{9} (4 + p)$

49. Word Play   An *anagram* is a word or phrase that is made by rearranging the letters of another word or phrase. Consider the word "SILENT."

   a. How many different arrangements of the letters in this word are possible? Write this number as a factorial. (*Hint:* First solve a simpler problem to see a pattern, such as how many different arrangements are there of just 2 letters? 3 letters?)

   b. If a friend gives you a hint and tells you that an anagram of this word starts with "L," how many different arrangements still remain?

   c. Your friend gives you one more hint. The last letter in the anagram is "N." Determine how many possible arrangements remain and then determine the anagram your friend is suggesting.

50. Chess   A standard chess board contains 64 small black or white squares. These squares make up many other larger squares of various sizes.
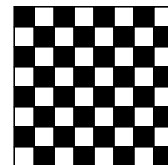
   a. How many 8×8 squares are there on a standard 8×8 chessboard? How many 7×7 squares?

   b. Continue this list until you have accounted for all 8 sizes of squares.

   c. Use sigma notation to represent the total number of squares found on an 8×8 chessboard. Then calculate this sum.

51. Use the comparison test to determine whether the series $\dfrac{3}{3} + \dfrac{3}{4} + \dfrac{3}{5} + \dfrac{3}{6} + \cdots$ is convergent or divergent. *(Lesson 12-4)*

52. Chemistry   A vacuum pump removes 20% of the air in a sealed jar on each stroke of its piston. The jar contains 21 liters of air before the pump starts. After how many strokes will only 42% of the air remain? *(Lesson 12-3)*

53. Find the first four terms of the geometric sequence for which $a_5 = 32\sqrt{2}$ and $r = -\sqrt{2}$. *(Lesson 12-2)*

54. Evaluate $\log_{10} 0.001$. *(Lesson 11-4)*

55. Write the standard form of the equation of the circle that passes through points at $(0, 9)$, $(-7, 2)$, and $(0, -5)$. *(Lesson 10-2)*

56. Simplify $\left(\sqrt{2} + i\right)\left(4\sqrt{2} + i\right)$. *(Lesson 9-5)*

57. **Sports** Find the initial vertical and horizontal velocities of a javelin thrown with an initial velocity of 59 feet per second at an angle of $63°$ with the horizontal. *(Lesson 8-7)*

58. Find the equation of the line that bisects the obtuse angle formed by the graphs of $2x - 3y + 9 = 0$ and $x + 4y + 4 = 0$. *(Lesson 7-7)*

59. **SAT/ACT Practice**   If $\dfrac{5 + m}{9 + m} = \dfrac{2}{3}$, then $m = ?$

A 8   **B** 6   **C** 5   **D** 3   **E** 2

CAREER CHOICES

Operations Research Analyst

In the changing economy of today, it is difficult to start and maintain a successful business. A business operator needs to be sure that the income from the business exceeds the expenses. Sometimes, businesses need the services of an operations research analyst. If you enjoy mathematics and solving tough problems, then you may want to consider a career as an operations research analyst.

In this occupation, you would gather many types of data about a business and analyze that data using mathematics and statistics. Examples of ways you might assist a business are: help a retail store determine the best store layout, help a bank in processing deposits more efficiently, or help a business set prices. Most operations research analysts work for private industry, private consulting firms, or the government.

CAREER OVERVIEW

Degree Preferred:
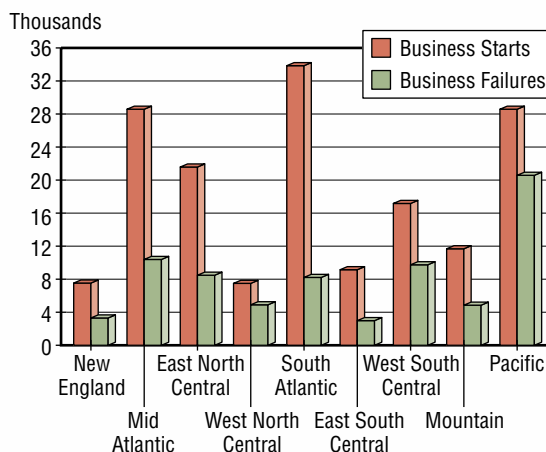bachelor's degree in applied mathematics

Related Courses:
mathematics, statistics, computer science, English

Outlook:
faster than average through the year 2006

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12-6

The Binomial Theorem

OBJECTIVE
- Use the Binomial Theorem to expand binomials.

Real World Application

FAMILY On November 19, 1997, Bobbie McCaughey, an Iowa seamstress, gave birth by Caesarian section to seven babies. The birth of the septuplets was the first of its kind in the United States since 1985. The babies, born after just 30 weeks of pregnancy, weighed from 2 pounds 5 ounces to 3 pounds 4 ounces. *A problem related to this will be solved in Example 2.*

Recall that a binomial, such as $x + y$, is an algebraic expression involving the sum of two unlike terms, in this case x and y. Just as there are patterns in sequences and series, there are numerical patterns in the expansion of powers of binomials. Let's examine the expansion of $(x + y)^n$ for $n = 0$ to $n = 5$. You already know a few of these expansions and the others can be obtained using algebraic properties.

$$(x + y)^0 = 1x^0y^0$$
$$(x + y)^1 = 1x^1y^0 + 1x^0y^1$$
$$(x + y)^2 = 1x^2y^0 + 2x^1y^1 + 1x^0y^2$$
$$(x + y)^3 = 1x^3y^0 + 3x^2y^1 + 3x^1y^2 + 1x^0y^3$$
$$(x + y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$$
$$(x + y)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5$$

Patterns in the Binomial Expansion of $(x + y)^n$

The following patterns can be observed in the expansion of $(x + y)^n$.
1. The expansion of $(x + y)^n$ has $n + 1$ terms.
2. The first term is x^n, and the last term is y^n.
3. In successive terms, the exponent of x decreases by 1, and the exponent of y increases by 1.
4. The degree of each term (the sum of the exponents of the variables) is n.
5. In any term, if the coefficient is multiplied by the exponent of x and the product is divided by the number of that term, the result is the coefficient of the following term.
6. The coefficients are symmetric. That is, the first term and the last term have the same coefficient. The second term and the second from the last term have the same coefficient, and so on.

If just the coefficients of these expansions are extracted and arranged in a triangular array, they form a pattern called **Pascal's triangle.** As you examine the triangle shown below, note that if two consecutive numbers in any row are added, the sum is a number in the following row. These three numbers also form a triangle.
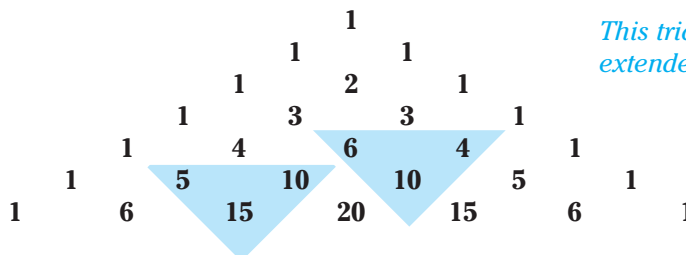
This triangle may be extended indefinitely.

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

Example **1** **Use Pascal's triangle to expand each binomial.**

a. $(x + y)^6$

First, write the series without the coefficients. Recall that the expression should have $6 + 1$ or 7 terms, with the first term being x^6 and the last term being y^6. Also note that the exponents of x should decrease from 6 to 0 while the exponents of y should increase from 0 to 6, while the degree of each term is 6.

$$x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6 \quad y^0 = 1, \, x^0 = 1$$

Then, use the numbers in the seventh row of Pascal's triangle as the coefficients of the terms. *Why is the seventh row used instead of the sixth row?*

$$\begin{array}{ccccccc} 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

b. $(3x + 2y)^7$

Extend Pascal's triangle to the eighth row.

$$\begin{array}{cccccccc} & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{array}$$

Then, write the expression and simplify each term. Replace each x with $3x$ and y with $2y$.

$$(3x + 2y)^7 = (3x)^7 + 7(3x)^6(2y) + 21(3x)^5(2y)^2 + 35(3x)^4(2y)^3 + 35(3x)^3(2y)^4 +$$
$$21(3x)^2(2y)^5 + 7(3x)(2y)^6 + (2y)^7$$

$$= 2187x^7 + 10{,}206x^6y + 20{,}412x^5y^2 + 22{,}680x^4y^3 + 15{,}120x^3y^4 +$$
$$6048x^2y^5 + 1344xy^6 + 128y^7$$

You can use Pascal's triangle to solve real-world problems in which there are only two outcomes for each event. For example, you can determine the distribution of answers on true-false tests, the combinations of heads and tails when tossing a coin, or the possible sequences of boys and girls in a family.

Example **2** **FAMILY** **Refer to the application at the beginning of the lesson. Of the seven children born to the McCaughey's, at least three were boys. How many of the possible groups of boys and girls have at least three boys?**

Let g represent girls and b represent boys. To find the number of possible groups, expand $(g + b)^7$. Use the eighth row of Pascal's triangle for the expansion.

$$g^7 + 7g^6b + 21g^5b^2 + 35g^4b^3 + 35g^3b^4 + 21g^2b^5 + 7gb^6 + b^7$$

To have at least three boys means that there could be 3, 4, 5, 6, or 7 boys. The total number of ways to have at least three boys is the same as the sum of the coefficients of g^4b^3, g^3b^4, g^2b^5, gb^6, and b^7. This sum is $35 + 35 + 21 + 7 + 1$ or 99.

Thus, there are 99 possible groups of boys and girls in which there are at least three boys.

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The general expansion of $(x + y)^n$ can also be determined by the **Binomial Theorem.**

Binomial Theorem	If n is a positive integer, then the following is true. $(x + y)^n = x^n + nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots + y^n$

Example ❸ Use the Binomial Theorem to expand $(2x - y)^6$.

The expansion will have seven terms. Find the first four terms using the sequence 1, $\dfrac{6}{1}$ or 6, $\dfrac{6\cdot 5}{1\cdot 2}$ or 15, $\dfrac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}$ or 20. Then use symmetry to find the remaining terms, 15, 6, and 1.

$(2x - y)^6 = (2x)^6 + 6(2x)^5(-y) + 15(2x)^4(-y)^2 + 20(2x)^3(-y)^3 + 15(2x)^2(-y)^4 + 6(2x)(-y)^5 + (-y)^6$

$= 64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6$

An equivalent form of the Binomial Theorem uses both sigma and factorial notation. It is written as follows, where n is a positive integer and r is a positive integer or zero.

$$(x + y)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} x^{n-r}y^r$$

You can use this form of the Binomial Theorem to find individual terms of an expansion.

Example ❹ Find the fifth term of $(4a + 3b)^7$.

$$(4a + 3b)^7 = \sum_{r=0}^{7} \frac{7!}{r!(7-r)!}(4a)^{7-r}(3b)^r$$

To find the fifth term, evaluate the general term for $r = 4$. *Since r increases from 0 to n, r is one less than the number of the term.*

$\dfrac{7!}{r!(7-r)!}(4a)^{7-r}(3b)^r = \dfrac{7!}{4!(7-4)!}(4a)^{7-4}(3b)^4$

$= \dfrac{7\cdot 6\cdot 5\cdot 4!}{4!\ 3!}(4a)^3(3b)^4$

$= 181{,}440a^3b^4$

The fifth term of $(4a + 3b)^7$ is $181{,}440a^3b^4$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. a. **Calculate** the sum of the numbers in each row of Pascal's triangle for $n = 0$ to 5.
   b. **Write** an expression to represent the sum of the numbers in the nth row of Pascal's triangle.

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2. Examine the expansion of $(x - y)^n$ for $n = 3, 4,$ and 5.

    a. Identify the sign of the second term for each expansion.

    b. Identify the sign of the third term for each expansion.

    c. Explain how to determine the sign of a term without writing out the entire expansion.

3. Restate what is meant by the observation that each term in a binomial expansion has degree n.

4. If ax^7y^b is a term from the expansion of $(x + y)^{12}$, **describe** how to determine its coefficient a and missing exponent b without writing the entire expansion.

Guided Practice

5. Use Pascal's triangle to expand $(c + d)^5$.

Use the Binomial Theorem to expand each binomial.

    6. $(a + 3)^6$ **7.** $(5 - y)^3$ **8.** $(3p - 2q)^4$

Find the designated term of each binomial expansion.

    9. 6th term of $(a - b)^7$

  10. 4th term of $\left(x + \sqrt{3}\right)^9$

  11. Coins A coin is flipped five times. Find the number of possible sets of heads and tails that have each of the following.

    a. 0 heads

    b. 2 heads

    c. at least 4 heads

    d. at most 3 heads

EXERCISES

Practice

Use Pascal's triangle to expand each binomial.

  12. $(a + b)^8$ **13.** $(n - 4)^6$ **14.** $(3c - d)^4$

15. Expand $(2 + a)^9$ using Pascal's triangle.

Use the Binomial Theorem to expand each binomial.

  16. $(d + 2)^7$ **17.** $(3 - x)^5$ **18.** $(4a + b)^4$

  19. $(2x - 3y)^3$ **20.** $\left(3m + \sqrt{2}\right)^4$ **21.** $\left(\sqrt{c} - 1\right)^6$

  22. $\left(\frac{1}{2}n + 2\right)^5$ **23.** $\left(3a + \frac{2}{3}b\right)^4$ **24.** $(p^2 + q)^8$

25. Expand $(xy - 2z^3)^6$ using the Binomial Theorem.

Find the designated term of each binomial expansion.

  26. 5th term of $(x + y)^9$ **27.** 4th term of $\left(a - \sqrt{2}\right)^8$

  28. 4th term of $(2a - b)^7$ **29.** 7th term of $(3c + 2d)^9$

  30. 8th term of $\left(\frac{1}{2}x - y\right)^{10}$ **31.** 6th term of $(2p - 3q)^{11}$

32. Find the middle term of the expansion of $\left(\sqrt{x} + \sqrt{y}\right)^8$.

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33. Business   A company decides to form a recycling committee to find a more efficient means of recycling waste paper. The committee is to be composed of eight employees. Of these eight employees, at least four women are to be on the committee. How many of the possible groups of men and women have at least four women?

34. Critical Thinking   Describe a strategy that uses the Binomial Theorem to expand $(a + b + c)^{12}$.

35. Education   Rafael is taking a test that contains a section of 12 true-false questions.
   a. How many of the possible groups of answers to these questions have exactly 8 correct answers of false?
   b. How many of the possible groups of answers to these questions have at least 6 correct answers of true?

36. Critical Thinking   Find a term in the expansion of $\left(3x^2 - \dfrac{1}{4x}\right)^6$ that does not contain the variable x.

37. Numerical Analysis   Before the invention of modern calculators and computers, mathematicians searched for ways to shorten lengthy calculations such as $(1.01)^4$.
   a. Express 1.01 as a binomial.
   b. Use the binomial representation of 1.01 found in part **a** and the Binomial Theorem to calculate the value of $(1.01)^4$ to eight decimal places.
   c. Use a calculator to estimate $(1.01)^4$ to eight decimal places. Compare this value to the value found in part **b**.

Mixed Review

38. Write $\displaystyle\sum_{k=2}^{7} 5 - 2k$ in expanded form and then find the sum. *(Lesson 12-5)*

39. Use the ratio test to determine whether the series $2 + \dfrac{2^2}{2!} + \dfrac{2^3}{3!} + \dfrac{2^4}{4!} + \cdots$ is *convergent* or *divergent*. *(Lesson 12-4)*

40. Find the sum of $\dfrac{2}{3} + \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{12} + \cdots$ or explain why one does not exist. *(Lesson 12-3)*

41. Finance   A bank offers a home mortgage for an annual interest rate of 8%. If a family decides to mortgage a $150,000 home over 30 years with this bank, what will the monthly payment for the principal and interest on their mortgage be? *(Lesson 11-2)*

42. Write $\overrightarrow{MK}$ as the sum of unit vectors if $M(-2, 6, 3)$ and $K(4, 8, -2)$. *(Lesson 8-3)*

43. Construction   A highway curve, in the shape of an arc of a circle, is 0.25 mile. The direction of the highway changes 45° from one end of the curve to the other. Find the radius of the circle in feet that the curve follows. *(Lesson 6-1)*
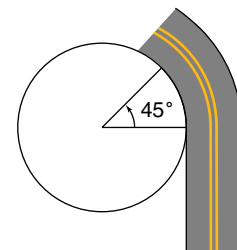
44. SAT/ACT Practice   If b is a prime integer such that $3b > 10 > \dfrac{5}{6}b$, which of the following is a possible value of b?

   A 2   **B** 3   **C** 4   **D** 11   **E** 13

Special Sequences and Series

Real World Application

NATURE An important sequence found in nature can be seen in the spiral of a Nautilus shell. To see this sequence follow procedure below.

- Begin by placing two small squares with side length 1 next to each other.
- Below both of these, draw a square with side length 2.
- Now draw a square that touches both a unit square and the 2-square. This square will have sides 3 units long.
- Add another square that touches both the 2-square and the 3-square. This square will have sides of 5 units.
- Continue this pattern around the picture so that each new square has sides that are as long as the sum of the latest two square's sides.

The side lengths of these squares form what is known as the **Fibonacci sequence:** 1, 1, 2, 3, 5, 8, 13, …. A spiral that closely models the Nautilus spiral can be drawn by first rearranging the squares so that the unit squares are in the interior and then connecting quarter circles, each of whose radius is a side of a new square. This spiral is known as the *Fibonacci spiral.* *A problem related to the Fibonacci sequence will be solved in Example 1.*
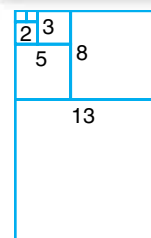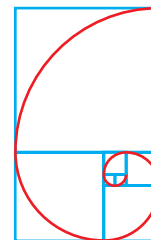
The Fibonacci sequence describes many patterns of numbers found in nature. This sequence was presented by an Italian mathematician named Leonardo de Pisa, also known as Fibonacci (pronounced *fih buh NACH ee*), in 1201. The first two numbers in the sequence are 1, that is, $a_1 = 1$ and $a_2 = 1$. As you have seen in the example above, adding the two previous terms generates each additional term in the sequence.

The original problem that Fibonacci investigated in 1202 involved the reproductive habits of rabbits under ideal conditions.

Example **1** **NATURE** **Suppose a newly born pair of rabbits are allowed to breed in a controlled environment. How many rabbit pairs will there be after one year if the following assumptions are made?**

Real World Application

- **A male and female rabbit can mate at the age of one month.**
- **At the end of its second month, a female rabbit can produce another pair of rabbits (one male, one female).**
- **The rabbits never die.**
- **The female always produces one new pair every month from the second month on.**

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Based on these assumptions, at the end of the first month, there will be one pair of rabbits. And at the end of the second month, the female produces a new pair, so now there are two pairs of rabbits.
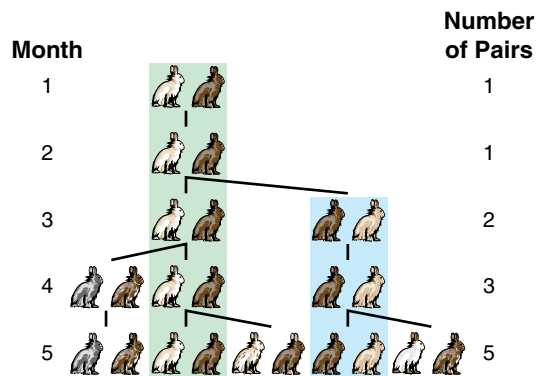
The following table shows the pattern.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Number of Pairs	1	1	2	3	5	8	13	21	34	55	89	144

There will be 144 pairs of rabbits during the twelfth month.

Each term in the sequence is the sum of the two previous terms.

Another important series is the series that is used to define the irrational number e. The Swiss mathematician Leonhard Euler (pronounced OY ler), published a work in which he developed this irrational number. It has been suggested that in his honor the number is called e, the Euler number. The number can be expressed as the sum of the following infinite series.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!} + \cdots$$

The Binomial Theorem can be used to derive the series for e as follows. Let k be any positive integer and apply the Binomial Theorem to $\left(1 + \frac{1}{k}\right)^k$.

$$\left(1 + \frac{1}{k}\right)^k = 1 + k\left(\frac{1}{k}\right) + \frac{k(k-1)}{2!}\left(\frac{1}{k}\right)^2 + \frac{k(k-1)(k-2)}{3!}\left(\frac{1}{k}\right)^3 + \cdots$$

$$+ \frac{k(k-1)(k-2)\cdots 1}{k!}\left(\frac{1}{k}\right)^k$$

$$= 1 + 1 + \frac{1\left(1-\frac{1}{k}\right)}{2!} + \frac{1\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)}{3!} + \cdots + \frac{1\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdots\frac{1}{k}}{k!}$$

Then, find the limit of $\left(1 + \frac{1}{k}\right)^k$ as k increases without bound.

$$\lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

As $k \to \infty$, the number of terms in the sum becomes infinite.

Thus, e can be defined as follows.

$$e = \lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k \quad \text{or} \quad e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

The value of e^x can be approximated by using the following series. This series is often called the **exponential series.**

Exponential Series

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Example ② **Use the first five terms of the exponential series and a calculator to approximate the value of $e^{2.03}$ to the nearest hundredth.**

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{2.03} \approx 1 + 2.03 + \frac{(2.03)^2}{2!} + \frac{(2.03)^3}{3!} + \frac{(2.03)^4}{4!}$$

$$\approx 1 + 2.03 + 2.06045 + 1.394237833 + 0.7075757004$$

$$\approx 7.19$$

Euler's name is associated with a number of important mathematical relationships. Among them is the relationship between the exponential series and a series called the **trigonometric series.** The trigonometric series for $\cos x$ and $\sin x$ are given below.

Trigonometric Series

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

These two series are convergent for all values of x. By replacing x with any angle measure expressed in radians and carrying out the computations, approximate values of the trigonometric functions can be found to any desired degree of accuracy.

Example 3 Use the first five terms of the trigonometric series to approximate the value of $\cos \frac{\pi}{3}$ to four decimal places.

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$\cos \frac{\pi}{3} \approx 1 - \frac{(1.0472)^2}{2!} + \frac{(1.0472)^4}{4!} - \frac{(1.0472)^6}{6!} + \frac{(1.0472)^8}{8!} \quad x = \frac{\pi}{3} \text{ or about } 1.0472$$
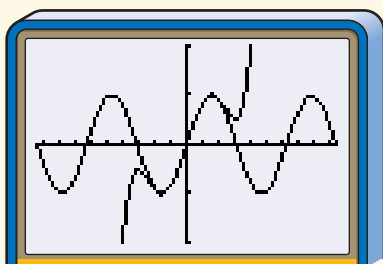
$$\cos \frac{\pi}{3} \approx 1 - 0.54831 + 0.05011 - 0.00183 + 0.00004$$

$$\cos \frac{\pi}{3} \approx 0.5004 \quad \textit{Compare this result to the actual value.}$$

GRAPHING CALCULATOR EXPLORATION

In this Exploration you will examine polynomial functions that can be used to approximate $\sin x$. The graph below shows the graphs of $f(x) = \sin x$ and $g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ on the same screen.

$[-3\pi, 3\pi]$ scl:1 by $[-2, 2]$ scl:1

TRY THESE

1. Use ⬚TRACE⬚ to help you write an inequality describing the x-values for which the graphs seem very close together.

2. In absolute value, what are the greatest and least differences between the values of $f(x)$ and $g(x)$ for the values of x described by the inequality you wrote in Exercise 1?

3. Repeat Exercises 1 and 2 using $h(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ instead of $g(x)$.

4. Repeat Exercises 1 and 2 using $k(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$.

WHAT DO YOU THINK?

5. Are the intervals for which you get good approximations for $\sin x$ larger or smaller for polynomials that have more terms?

6. What term should be added to $k(x)$ to obtain a polynomial with six terms that gives good approximations to $\sin x$?

Another very important formula is derived by replacing x by $i\alpha$ in the exponential series, where i is the imaginary unit and α is the measure of an angle in radians.

$$e^{i\alpha} = 1 + i\alpha + \frac{(i\alpha)^2}{2!} + \frac{(i\alpha)^3}{3!} + \frac{(i\alpha)^4}{4!} + \cdots$$

$$e^{i\alpha} = 1 + i\alpha - \frac{\alpha^2}{2!} - i\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \cdots \quad i^2 = -1, \, i^3 = -i, \, i^4 = 1$$

Group the terms according to whether they contain the factor i.

$$e^{i\alpha} = \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots\right) + i\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots\right)$$

Notice that the real part is exactly $\cos \alpha$ and the imaginary part is exactly $\sin \alpha$. This relationship is called **Euler's Formula.**

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<table>
<tr><td>Euler's Formula</td><td>$e^{i\alpha} = \cos \alpha + i \sin \alpha$</td></tr>
</table>

If $-i\alpha$ had been substituted for x rather than $i\alpha$, the result would have been $e^{-i\alpha} = \cos \alpha - i \sin \alpha$.

Euler's Formula can be used to write a complex number, $a + bi$, in its exponential form, $re^{i\theta}$.

$$a + bi = r(\cos \theta + i \sin \theta)$$
$$= re^{i\theta}$$

Example **4** Write $1 + \sqrt{3}i$ in exponential form.

Write the polar form of $1 + \sqrt{3}i$. Recall that $a + bi = r(\cos \theta + i \sin \theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = \text{Arctan} \dfrac{b}{a}$ when $a > 0$.

$r = \sqrt{(1)^2 + (\sqrt{3})^2}$ or 2, and $\theta = \text{Arctan} \dfrac{\sqrt{3}}{1}$ or $\dfrac{\pi}{3}$ *a = 1 and b = $\sqrt{3}$*

$$1 + \sqrt{3}i = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$
$$= 2e^{i\frac{\pi}{3}}$$

Thus, the exponential form of $1 + \sqrt{3}i$ is $2e^{i\frac{\pi}{3}}$.

The equations for $e^{i\alpha}$ and $e^{-i\alpha}$ can be used to derive the exponential values of $\sin \alpha$ and $\cos \alpha$.

$$e^{i\alpha} - e^{-i\alpha} = (\cos \alpha + i \sin \alpha) - (\cos \alpha - i \sin \alpha)$$
$$e^{i\alpha} - e^{-i\alpha} = 2i \sin \alpha$$
$$\sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

$$e^{i\alpha} + e^{-i\alpha} = (\cos \alpha + i \sin \alpha) + (\cos \alpha - i \sin \alpha)$$
$$e^{i\alpha} + e^{-i\alpha} = 2 \cos \alpha$$
$$\cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

From your study of logarithms, you know that there is no real number that is the logarithm of a negative number. However, you can use a special case of Euler's Formula to find a complex number that is the natural logarithm of a negative number.

$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$
$$e^{i\pi} = \cos \pi + i \sin \pi \quad \textit{Let } \alpha = \pi.$$
$$e^{i\pi} = -1 + i(0)$$
$$e^{i\pi} = -1 \qquad\qquad \textit{So } e^{i\pi} + 1 = 0.$$

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If you take the natural logarithm of both sides of $e^{i\pi} = -1$, you can obtain a value for $\ln(-1)$.

$$\ln e^{i\pi} = \ln(-1)$$
$$i\pi = \ln(-1)$$

Thus, the natural logarithm of a negative number $-k$, for $k > 0$, can be defined using $\ln(-k) = \ln(-1)k$ or $\ln(-1) + \ln k$, a complex number.

Example **5** **Evaluate ln (−270).**

$\ln(-270) = \ln(-1) + \ln(270)$ *Use a calculator to compute ln (270).*
$\approx i\pi + 5.5984$

Thus, $\ln(-270) \approx i\pi + 5.5984$.   *The logarithm is a complex number.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why in Example 2 of this lesson the exponential series gives an approximation of 7.19 for $e^{2.03}$, while a calculator gives an approximation of 7.61.

2. **Estimate** for what values of x the series $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$ gives a good approximation of the function e^x using the graph at the right.

3. **List** at least two reasons why Fibonacci's rabbit problem in Example 1 is not realistic.

4. **Write** a recursive formula for the terms of the Fibonacci sequence.
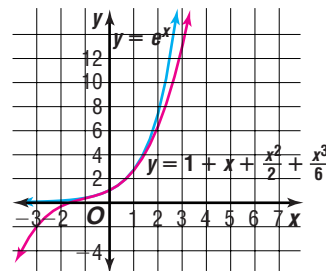
Guided Practice

Find each value to four decimal places.

5. $\ln(-7)$

6. $\ln(-0.379)$

Use the first five terms of the exponential series and a calculator to approximate each value to the nearest hundredth.

7. $e^{0.8}$

8. $e^{1.36}$

9. Use the first five terms of the trigonometric series to approximate the value of $\sin \pi$ to four decimal places. Then, compare the approximation to the actual value.

Write each complex number in exponential form.

10. $\sqrt{2}\left(\cos \dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4}\right)$

11. $-1 + \sqrt{3}i$

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12. Investment The Cyberbank advertises an Advantage Plus savings account with a 6% interest rate compounded continuously. Morgan is considering opening up an Advantage Plus account, because she needs to double the amount she will deposit over 5 years.

  a. If Morgan deposits P dollars into the account, approximate the return on her investment after 5 years using three terms of the exponential series. (*Hint:* The formula for continuous compounding interest is $A = Pe^{rt}$.)

  b. Will Morgan double her money in the desired amount of time? Explain.

  c. Check the approximation found in part **a** using a calculator. How do the two answers compare?

EXERCISES

Practice

Find each value to four decimal places.

13. $\ln(-4)$ **14.** $\ln(-3.1)$ **15.** $\ln(-0.25)$

16. $\ln(-0.033)$ **17.** $\ln(-238)$ **18.** $\ln(-1207)$

Use the first five terms of the exponential series and a calculator to approximate each value to the nearest hundredth.

19. $e^{1.1}$ **20.** $e^{-0.2}$ **21.** $e^{4.2}$

22. $e^{0.55}$ **23.** $e^{3.5}$ **24.** $e^{2.73}$

Use the first five terms of the trigonometric series to approximate the value of each function to four decimal places. Then, compare the approximation to the actual value.

25. $\cos \pi$ **26.** $\sin \dfrac{\pi}{4}$ **27.** $\cos \dfrac{\pi}{6}$

28. Approximate the value of $\sin \dfrac{\pi}{2}$ to four decimal places using the first five terms of the trigonometric series.

Write each complex number in exponential form.

29. $5\left(\cos \dfrac{5\pi}{3} + i \sin \dfrac{5\pi}{3}\right)$ **30.** i

31. $1 + i$ **32.** $\sqrt{3} + i$

33. $-\sqrt{2} + \sqrt{2}i$ **34.** $-4\sqrt{3} - 4i$

35. Write the expression $3 + 3i$ in exponential form.

Applications and Problem Solving

36. Research Explore the meaning of the term *transcendental number*.

37. Critical Thinking Show that for all real numbers x, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$.

38. Research Investigate and write a one-page paper on the occurrence of Fibonacci numbers in plants.

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39. Critical Thinking Examine Pascal's triangle. What relationship can you find between Pascal's triangle and the Fibonacci sequence?

40. Number Theory Consider the Fibonacci sequence 1, 1, 2, 3, …, $F_{n-2} + F_{n-1}$.

a. Find $\dfrac{F_n}{F_{n-1}}$ for the second through eleventh terms of the Fibonacci sequence.

b. Is this sequence of ratios arithmetic, geometric, or neither?

c. Sketch a graph of the terms found in part **a**. Let n be the x-coordinate and $\dfrac{F_n}{F_{n-1}}$ be the y-coordinate, and connect the points.

d. Based on the graph found in part c, does this sequence appear to approach a limit? If so, use the last term found to approximate this limit to three decimal places.

e. The *golden ratio* has a value of approximately 1.61804. How does the limit of the sequence $\dfrac{F_n}{F_{n-1}}$ compare to the golden ratio?

f. Research the term *golden ratio*. Write several paragraphs on the history of the golden ratio and describe its application to art and architecture.

41. Investment When Cleavon turned 5 years old, his grandmother decided it was time to start saving for his college education. She deposited $5000 in a special account earning 5% interest compounded continuously. By the time Cleavon begins college at the age of 18, his grandmother estimates that her grandson will need $40,000 for college tuition.

a. Approximate Cleavon's savings account balance on his 18th birthday using five terms of the exponential series.

b. Will the account have sufficient funds for Cleavon's college tuition by the time he is ready to start college? Explain.

c. Use a calculator to compute how long it will take the account to accumulate $40,000. How old would Cleavon be by this time?

d. To have at least $40,000 when Cleavon is 18 years old, how much should his grandmother have invested when he was 5 years old to the nearest dollar?

42. Critical Thinking Find a pattern, if one exists, for the following types of Fibonacci numbers.

a. even numbers

b. multiples of 3

CONTENTS

43. Use the Binomial Theorem to expand $(2x + y)^6$. *(Lesson 12-6)*

44. Express the series $2 + 4 + 8 + \cdots + 64$ using sigma notation. *(Lesson 12-5)*

45. Number Theory   If you tear a piece of paper that is 0.005 centimeter thick in half, and place the two pieces on top of each other, the height of the pile of paper is 0.01 centimeter. Let's call this the second pile. If you tear these two pieces of paper in half, the third pile will have four pieces of paper in it. *(Lesson 12-2)*

   a. How high is the third pile? the fourth pile?

   b. Write a formula to determine how high the nth pile is.

   c. Use the formula to determine in theory how high the 10th pile would be and how high the 100th pile would be.

46. Evaluate $\dfrac{8^{\frac{2}{3}}}{8^{\frac{1}{3}}}$. *(Lesson 11-1)*

47. Write an equation of the parabola in general form that has a horizontal axis and passes through points at $(0, 0)$, $(2, -1)$, and $(4, -4)$. *(Lesson 10-5)*

48. Find the quotient of $16\left(\cos \dfrac{\pi}{8} + i \sin \dfrac{\pi}{8}\right)$ divided by $4\left(\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right)$. *(Lesson 9-7)*

49. Physics   Two forces, one of 30 N and the other of 50 N, act on an object. If the angle between the forces is $40°$, find the magnitude and the direction of the resultant force. *(Lesson 8-5)*
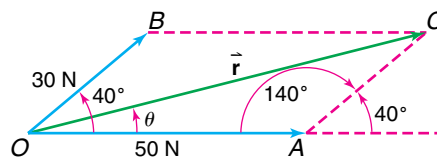
50. Carpentry   Carpenters use circular sanders to smooth rough surfaces, such as wood or plaster. The disk of a sander has a radius of 6 inches and is rotating at a speed of 5 revolutions per second. *(Lesson 6-2)*

   a. Find the angular velocity of the sander disk in radians per second.

   b. What is the linear velocity of a point on the edge of the sander disk in feet per second?

51. Education   You may answer up to 30 questions on your final exam in history class. It consists of multiple-choice and essay questions. Two 48-minute class periods have been set aside for taking the test. It will take you 1 minute to answer each multiple-choice question and 12 minutes for each essay question. Correct answers of multiple-choice questions earn 5 points and correct essay answers earn 20 points. If you are confident that you will answer all of the questions you attempt correctly, how many of each type of questions should you answer to receive the highest score? *(Lesson 2-7)*

52. SAT/ACT Practice   In triangle ABC, if $BD = 5$, what is the length of $\overline{BC}$?

   A 3

   B 5

   C $\sqrt{39}$

   D $\sqrt{70}$

   E $\sqrt{126}$

Extra Practice   See p. A50.

12-8
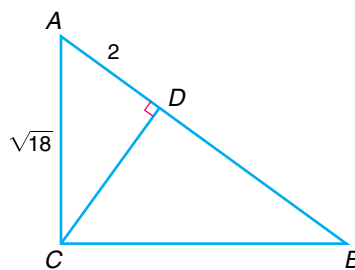
Sequences and Iteration

OBJECTIVES
- Iterate functions using real and complex numbers.

ECOLOGY The population of grizzly bears on the high Rocky Mountain Front near Choteau, Montana, has a growth factor of 1.75. The maximum population of bears that can be sustained in the area is 500 bears, and the current population is 240. Write an equation to model the population. Use the equation to find the population of bears at the end of fifteen years. *This problem will be solved in Example 1.*

The population of a species in a defined area changes over time. Changes in the availability of food, good or bad weather conditions, the amount of hunting allowed, disease, and the presence or absence of predators can all affect the population of a species. We can use a mathematical equation to model the changes in a population.

One such model is the *Verhulst population model.* The model uses the recursive formula $p_{n+1} = p_n + rp_n(1 - p_n)$, where n represents the number of time periods that have passed, p_n represents the percent of the maximum sustainable population that exists at time n, and r is a growth factor.

Example ① **ECOLOGY** **Refer to the application above.**

a. Write an equation to model the population.

For this problem, the Verhulst population model can be used to represent the changes in the population of a species.

$p_{n+1} = p_n + rp_n(1 - p_n)$
$p_{n+1} = p_n + 1.75p_n(1 - p_n)$ *r = 1.75*

b. Find the population of bears at the end of fifteen years.

The initial percent of the maximum sustainable population can be represented by the ratio $p_0 = \dfrac{240}{500}$ or 0.48. This means that the current population is 48% of the maximum sustainable population.

Now, we can find the first few iterates as follows.

$p_1 = 0.48 + 1.75(0.48)(1 - 0.48)$ or 0.9168   *n = 1, p₀ = 0.48,*
(0.9168)(500) ≈ 458 bears   *r = 1.75*

$p_2 = 0.9168 + 1.75(0.9168)(1 - 0.9168)$ or 1.0503   *n = 2, p₁ = 0.9168,*
(1.0503)(500) ≈ 525 bears   *r = 1.75*

$p_3 = 1.0503 + 1.75(1.0503)(1 - 1.0503)$ or 0.9578   *n = 3, p₂ = 1.0503,*
(0.9578)(500) ≈ 479 bears   *r = 1.75*

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The table shows the remainder of the values for the first fifteen years.

n	4	5	6	7	8	9	10	11	12	13	14	15
number of bears	514	489	508	494	505	497	503	498	501	499	501	499

At the end of the first 15 years, there could be 499 bears.

Notice that to determine the percent of the maximum sustainable population for a year, you must use the percent from the previous year. In Lesson 1-2, you learned that this process of composing a function with itself repeatedly is called *iteration*. Each output is called an *iterate*. To iterate a function $f(x)$, find the function value $f(x_0)$ of the initial value x_0. The second iterate is the value of the function performed on the output, that is $f(f(x_0))$.

Example ② **BANKING** **Selina Anthony has a savings account that has an annual yield of 6.3%. Find the balance of the account after each of the first three years if her initial balance is \$4210.**

The balance of a savings account, using simple interest compounded at the end of a period of time, can be found by iterating the function $p_{n+1} = p_n + rp_n$, where p_n is the principal after n periods of time and r is the interest rate for a period of time.

$p_1 = p_0 + rp_0$
$p_1 = 4210 + (0.063)(4210)$ *$p_0 = 4210, r = 0.063$*
$p_1 = \$4475.23$

$p_2 = 4475.23 + (0.063)(4475.23)$ *$p_1 = 4475.23, r = 0.063$*
$p_2 \approx \$4757.17$

$p_3 = 4757.17 + (0.063)(4757.17)$ *$p_2 = 4757.17, r = 0.063$*
$p_3 \approx \$5056.87$

The balance in Ms. Anthony's account at the end of the three years is \$5056.87.

Look Back
You can review graphing numbers in the complex plane in Lesson 9-6

Remember that every complex number has a real part and an imaginary part. The complex number $a + bi$ has been graphed on the complex plane at the right. The horizontal axis of the complex plane represents the real part of the number, and the vertical axis represents the imaginary part.
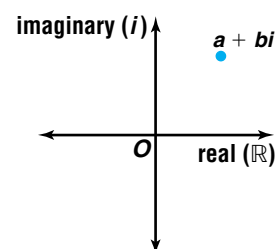
Functions can be iterated using the complex numbers as the domain.

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3 Find the first three iterates of the function $f(z) = 0.5z + 2$, if the initial value is $20 + 16i$.

f(z) is used to denote a function on the complex number plane.

$z_0 = 20 + 16i$
$z_1 = 0.5(20 + 16i) + 2$ or $12 + 8i$
$z_2 = 0.5(12 + 8i) + 2$ or $8 + 4i$
$z_3 = 0.5(8 + 4i) + 2$ or $6 + 2i$

We can graph this sequence of iterates on the complex plane. The graph at the right shows the **orbit,** or sequence of successive iterates, of the initial value of $z_0 = 20 + 16i$ from Example 3.
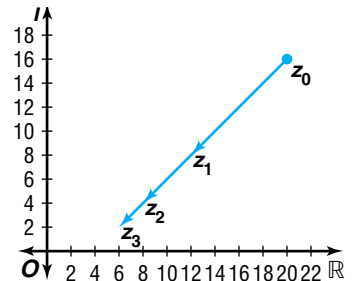
4 Consider the function $f(z) = z^2 + c$, where c and z are complex numbers.

a. Find the first six iterates of the function for $z_0 = 1 + i$ and $c = -i$.

b. Plot the orbit of the initial point at $1 + i$ under iteration of the function $f(z) = z^2 - i$ for six iterations.

c. Describe the long-term behavior of the function under iteration.

a. $z_1 = (1 + i)^2 - i$
$\quad = i$

$z_2 = (i)^2 - i$
$\quad = -1 - i$

$z_3 = (-1 - i)^2 - i$
$\quad = i$

$z_4 = (i)^2 - i$
$\quad = -1 - i$

$z_5 = (-1 - i)^2 - i$
$\quad = i$

$z_6 = (i)^2 - i$
$\quad = -1 - i$

b.
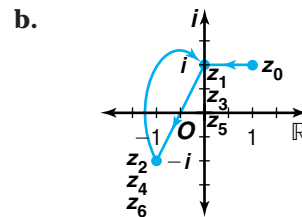
c. The iterates repeat every two iterations.

For many centuries, people have used science and mathematics to better understand the world. However, Euclidean geometry, the familiar geometry of points, lines, and planes, is not adequate to describe the natural world. Objects like coastlines, clouds, and mountain ranges can be more easily approximated by a new type of geometry. This type of geometry is called **fractal geometry.** The function $f(z) = z^2 + c$, where c and z are complex numbers is central to the study of fractal geometry.

At the heart of fractal geometry are the *Julia sets.* In Chapter 9, you learned that Julia sets involve graphing the behavior of a function that is iterated in the complex plane. As a function is iterated, the iterates either escape or are held prisoner. If the sequence of iterates remains within a finite distance from $0 + 0i$, the initial point is called a **prisoner point.** If the sequence of iterates does not remain within a finite distance of $0 + 0i$, the point is called an **escaping point.**

Suppose the function $f(z) = z^2 + c$ is iterated for two different initial values. If the first one is an escaping point and the second one is a prisoner point, the graphs of the orbits may look like those shown below.
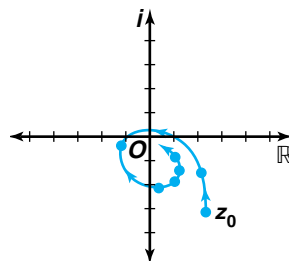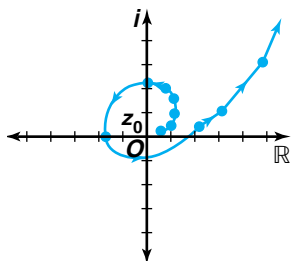
Example **5** Find the first four iterates of the function $f(z) = z^2 + c$ where $c = 0 + 0i$ for initial values whose moduli are in the regions $|z_0| < 1$, $|z_0| = 1$, and $|z_0| > 1$. Graph the orbits and describe the results.
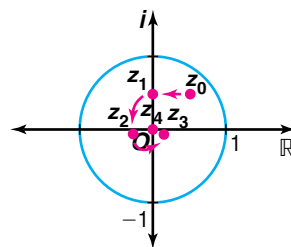
Recall from Lesson 9-6 that the modulus of a complex number is the distance that the graph of the complex number is from the origin. So $|a + bi| = \sqrt{a^2 + b^2}$.

Choose an initial value in each of the intervals. For $|z_0| < 1$, we will use $z_0 = 0.5 + 0.5i$; for $|z_0| = 1$, $0.5 + 0.5\sqrt{3}i$; and for $|z_0| > 1$, $0.75 + 0.75i$.
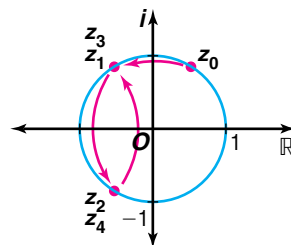
For $|z_0| < 1$ $z_0 = 0.5 + 0.5i$
$z_1 = (0.5 + 0.5i)^2$ or $0.5i$
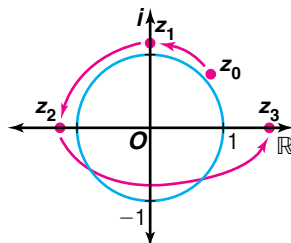$z_2 = (0.5i)^2$ or -0.25
$z_3 = (-0.25)^2$ or 0.0625
$z_4 = 0.0625^2$ or 0.0039

For $|z_0| = 1$ $z_0 = 0.5 + 0.5\sqrt{3}i$
$z_1 = -0.5 + 0.5\sqrt{3}i$
$z_2 = -0.5 - 0.5\sqrt{3}i$
$z_3 = -0.5 + 0.5\sqrt{3}i$
$z_4 = -0.5 - 0.5\sqrt{3}i$

For $|z_0| > 1$ $z_0 = 0.75 + 0.75i$
$z_1 = 1.125i$
$z_2 = -1.2656$
$z_3 = 1.602$
$z_4 = 2.566$

In the first case, where $|z_0| < 1$, the iterates will approach 0, since each successive square will be smaller than the one before. Thus, the orbit in this case approaches 0, a fixed point. In the second case, where $|z_0| = 1$, the iterates are orbiting around on the unit circle. When $|z_0| > 1$, the iterates approach infinity, since each square will be greater than the one before.

The orbits in Example 5 demonstrate that the iterative behavior of various initial values in different regions behave in different ways. An initial value whose graph is inside the unit circle is a prisoner point. A point chosen on the unit circle stays on the unit circle, and a point outside of the unit circle escapes. Other functions of the form $f(z) = z^2 + c$ when iterated over the complex plane also have regions in which the points behave this way; however, they are usually not circles.

All of the initial points for a function on the complex plane are split into three sets, those that escape (called the *escape set* E), those that do not escape (called the *prisoner set* P) and the boundary between the escape set and the prisoner set (the Julia set). The escape set, the prisoner set, and the Julia set for the function in Example 5 are graphed at the right. The Julia set in Example 5 is the unit circle.
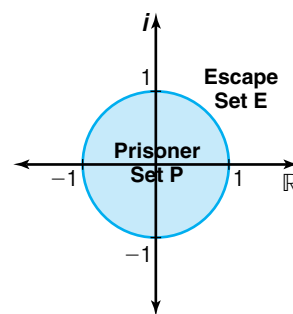
CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** how iteration and composition functions are related.

2. **Describe** the orbit of a complex number under iteration.

3. **Explain** how a Julia set of a function is related to the prisoner set of the function.

Guided Practice

Find the first four iterates of each function using the given initial value. If necessary, round answers to the nearest hundredth.

4. $f(x) = x^2; x_0 = -1$

5. $f(x) = 2x - 5; x_0 = 2$

Find the first three iterates of the function $f(z) = 0.6z + 2i$ for each initial value.

6. $z_0 = 6i$

7. $z_0 = 25 + 40i$

Find the first three iterates of the function $f(z) = z^2 + c$ for each given value of c and each initial value.

8. $c = 1 + 2i; z_0 = 0$

9. $c = 2 - 3i; z_0 = 1 + 2i$

10. **Biology** The Verhulst population model describing the population of antelope in an area is $p_{n+1} = p_n + 1.75p_n(1 - p_n)$. The maximum population sustainable in the area is 40, and the current population is 24. Find the population of antelope after each of the first ten years.

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Practice

Find the first four iterates of each function using the given initial value. If necessary, round answers to the nearest hundredth.

11. $f(x) = 3x - 7; x_0 = 4$ **12.** $f(x) = x^2; x_0 = -2$ **13.** $f(x) = (x - 5)^2; x_0 = 4$

14. $f(x) = x^2 - 1; x_0 = -1$ **15.** $f(x) = 2x^2 - x; x_0 = 0.1$

16. Find the first ten iterates of $f(x) = \dfrac{2}{x}$ for each initial value.

    a. $x_0 = 1$ **b.** $x_0 = 4$ **c.** $x_0 = 7$

    d. What do you observe about the iterates of this function?

Find the first three iterates of the function $f(z) = 2z + (3 - 2i)$ for each initial value.

17. $z_0 = 5i$ **18.** $z_0 = 4$ **19.** $z_0 = 1 + 2i$

20. $z_0 = -1 - 2i$ **21.** $z_0 = 6 + 2i$ **22.** $z_0 = 0.3 - i$

23. Find the first three iterates of the function $f(z) = 3z - 2i$ for $z_0 = \dfrac{1}{3} + \dfrac{2}{3}i$.

Find the first three iterates of the function $f(z) = z^2 + c$ for each given value of c and each initial value.

24. $c = -1; z_0 = 0 - i$ **25.** $c = 1 - 3i; z_0 = i$

26. $c = 3 + 2i; z_0 = 1$ **27.** $c = -4i; z_0 = 1 + i$

28. $c = 0; z_0 = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$ **29.** $c = 2 + 3i; z_0 = 1 - i$

**Applications and Problem Solving**

Real World Application

30. Banking Amelia has a savings account that has an annual yield of 5.2%. Find the balance of the account after each of the first five years if her initial balance is $2000.

31. Ecology The population of elk on the Bridger Range in the Rocky Mountains of western Montana has a growth factor of 2.5. The population in 1984 was 10% of the maximum population sustainable. What percent of the maximum sustainable population should be present in 2002?

32. Critical Thinking If $f(z) = z^2 + c$ is iterated with an initial value of $2 + 3i$ and $z_1 = -1 + 15i$, find c.

33. Critical Thinking In Exercise 16, find an initial value that produces iterates that all have the same value.

34. Research Investigate the applications of fractal geometry to agriculture. How is fractal geometry being used in this field? Write several paragraphs about your findings.

35. Critical Thinking

    a. Use a calculator to find $\sqrt{2}, \sqrt{\sqrt{2}}, \sqrt{\sqrt{\sqrt{2}}},$ and $\sqrt{\sqrt{\sqrt{\sqrt{2}}}}$.

    b. Define an iterative function $f(z)$ that models the situation described in part a.

    c. Determine the limit of $f(z)$ as the number of iterations approaches infinity.

    d. Determine the limit of $f(z)$ as the number of iterations approaches infinity for integral values of $z_0 > 0$.

36. Write $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ in exponential form. *(Lesson 12-7)*

37. Find the fifth term in the binomial expansion of $(2a - 3b)^8$. *(Lesson 12-6)*

38. State whether the series $\dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{32} + \cdots$ is *convergent* or *divergent*. Explain your reasoning. *(Lesson 12-4)*

39. Nuclear Power   A nuclear cooling tower has an eccentricity of $\dfrac{7}{5}$. At its narrowest point the cooling tower is 130 feet wide. Determine the equation of the hyperbola used to generate the hyperboloid of the cooling tower. *(Lesson 10-4)*

40. Optics   A beam of light strikes a diamond at an angle of 42°. What is the angle of refraction if the index of refraction of a diamond is 2.42 and the index of refraction of air is 1.00? (Use Snell's Law, $n_1 \sin I = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium it is entering, I is the angle of incidence, and r is the angle of refraction.) *(Lesson 7-5)*

41. Air Traffic Safety   The traffic pattern for airplanes into San Diego's airport is over the heart of downtown. Therefore, there are restrictions on the heights of new construction. The owner of an office building wishes to erect a microwave tower on top of the building. According to the architect's design, the angle of elevation from a point on the ground to the top of the 40-foot tower is 56°. The angle of elevation from the ground point to the top of the building is 42°. *(Lesson 5-4)*
a. Draw a sketch of this situation.
b. The maximum allowed height of any structure is 100 feet. Will the city allow the building of this tower? Explain.

42. Determine whether the graph has infinite discontinuity, jump discontinuity, or point discontinuity, or it is continuous. *(Lesson 3-5)*
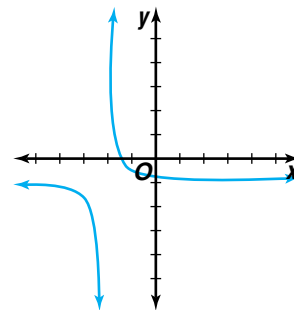
43. Find the maximum and minimum values of the function $f(x, y) = 2x + 8y + 10$ for the polygonal convex set determined by the following system of inequalities. *(Lesson 2-6)*
$x \geq 3$
$x \leq 8$
$5 \leq y \leq 9$
$x + y \leq 14$

44. SAT/ACT Practice   Two students took a science test and received different scores between 10 and 100. If H equals the higher score and L equals the lower score, and the difference between the two scores equals the average of the two scores, what is the value of $\dfrac{H}{L}$?

A $\dfrac{3}{2}$ **B** 2   **C** $\dfrac{5}{2}$ **D** 3

E It cannot be determined from the information given.

Mathematical Induction

Real World Application

BUSINESS Felipe and Emily work a booth on the midway at the state fair. Before the fair opens, they discover that their cash supply contains only $5 and $10 bills. Rene volunteers to go and get a supply of $1s, but before leaving, Rene remarks that she could have given change for any amount greater than $4 had their cash supply contained only $2 and $5 bills. Felipe replies, "Even if we had $2 bills, you would still need $1 bills." Rene disagrees and says that she can prove that she is correct. *This problem will be solved in Example 4.*

A method of proof called **mathematical induction** can be used to prove certain conjectures and formulas. Mathematical induction depends on a recursive process that works much like an unending line of dominoes arranged so that if any one domino falls the next one will also fall.

Suppose the first domino is knocked over.	Condition 1
The first will knock down the second.	Condition 2
The second will knock down the third.	Condition 3
The third will knock down the fourth.	Condition 4
$\vdots$	$\vdots$

Thus, the whole line of dominos will eventually fall.

Mathematical induction operates in a similar manner. If a statement S_k implies the truth of S_{k+1} and the statement S_1 is true, then the chain reaction follows like an infinite set of tumbling dominoes.

S_1 is true.	Condition 1
S_1 implies that S_2 is true.	Condition 2
S_2 implies that S_3 is true.	Condition 3
S_3 implies that S_4 is true.	Condition 4
$\vdots$	$\vdots$

In general, the following steps are used to prove a conjecture by mathematical induction.

Proof by Mathematical Induction

1. First, verify that the conjecture S_n is valid for the first possible case, usually $n = 1$. *This is called the anchor step.*
2. Then, assume that S_n is valid for $n = k$, and use this assumption to prove that it is also valid for $n = k + 1$. *This is called the induction step.*

Thus, since S_n is valid for $n = 1$ (or any other first case), it is valid for $n = 2$. Since it is valid for $n = 2$, it is valid for $n = 3$, and so on, indefinitely.

Mathematical induction can be used to prove summation formulas.

Example 1 **Prove that the sum of the first n positive integers is $\dfrac{n(n + 1)}{2}$.**

Here S_n is defined as $1 + 2 + 3 + \cdots + n = \dfrac{n(n + 1)}{2}$.

1. First, verify that S_n is valid for $n = 1$.

Since the first positive integer is 1 and $\dfrac{1(1 + 1)}{2} = 1$, the formula is valid for $n = 1$.

2. Then assume that S_n is valid for $n = k$.

$$S_k \Rightarrow 1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2} \quad \textit{Replace n with k.}$$

Next, prove that it is also valid for $n = k + 1$.   *Add $(k + 1)$ to*

$$S_{k + 1} \Rightarrow 1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \quad \textit{both sides of } S_k.$$

We can simplify the right side by adding $\dfrac{k(k + 1)}{2} + (k + 1)$.

$(k + 1)$ is a

$$S_{k + 1} \Rightarrow 1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1) + 2(k + 1)}{2} \quad \textit{common factor}$$
$$= \frac{(k + 1)(k + 2)}{2}$$

If $k + 1$ is substituted into the original formula $\left(\dfrac{n(n + 1)}{2}\right)$, the same result is obtained.
$$\frac{(k + 1)(k + 1 + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$$

Thus, if the formula is valid for $n = k$, it is also valid for $n = k + 1$. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. That is, the formula for the sum of the first n positive integers holds.

Mathematical induction is also used to prove properties of divisibility. Recall that an integer p is *divisible* by an integer q if $p = qr$ for some integer r.

Example 2 **Prove that $4^n - 1$ is divisible by 3 for all positive integers n.**

Using the definition of divisibility, we can state the conjecture as follows:

Notice that here S_n does not represent a sum but a conjecture.

$$S_n \Rightarrow 4^n - 1 = 3r \text{ for some integer } r$$

1. First verify that S_n is valid for $n = 1$.

$S_1 \Rightarrow 4^1 - 1 = 3$. Since 3 is divisible by 3, S_n is valid for $n = 1$.

2. Then assume that S_n is valid for $n = k$ and use this assumption to prove that it is also valid for $n = k + 1$.

$S_k \Rightarrow 4^k - 1 = 3r$ for some integer r *Assume S_k is true.*
$S_{k + 1} \Rightarrow 4^{k + 1} - 1 = 3t$ for some integer t *Show that $S_{k + 1}$ must follow.*

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For this proof, rewrite the left-hand side of S_k so that it matches the left-hand side of S_{k+1}.

$4^k - 1 = 3r$	S_k
$4(4^k - 1) = 4(3r)$	*Multiply each side by 4.*
$4^{k+1} - 4 = 12r$	*Simplify.*
$4^{k+1} - 1 = 12r + 3$	*Add 3 to each side.*
$4^{k+1} - 1 = 3(4r + 3)$	*Factor.*

Let $t = 4r + 3$, an integer. Then $4^{k+1} - 1 = 3t$

We have shown that if S_k is valid, then S_{k+1} is also valid. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. Hence, $4^n - 1$ is divisible by 3 for all positive integers n.

There is no "fixed" way of completing Step 2 for a proof by mathematical induction. Often, each problem has its own special characteristics that require a different technique to complete the proof. You may have to multiply the numerator and denominator of an expression by the same quantity, factor or expand an expression in a special way, or see an important relationship with the distributive property.

Example **3** **Prove that $6^n - 2^n$ is divisible by 4 for all positive integers n.**

Begin by restating the conjecture using the definition of divisibility.

$S_n \Rightarrow 6^n - 2^n = 4r$ for some integer r

1. Verify that S_n is valid for $n = 1$

   $S_1 \Rightarrow 6^1 - 2^1 = 4$. Since 4 is divisible by 4, S_n is valid for $n = 1$.

2. Assume that S_n is valid for $n = k$ and use this to prove that it is also valid for $n = k + 1$.

$S_k \Rightarrow 6^k - 2^k = 4r$ for some integer r	*Assume S_k is true*
$S_{k+1} \Rightarrow 6^{k+1} - 2^{k+1} = 4t$ for some integer t	*Show that S_{k+1} must follow.*

   Begin by rewriting S_k as $6^k = 2^k + 4r$. Now rewrite the left-hand side of this expression so that it matches the left-hand side of S_{k+1}.

$6^k = 2^k + 4r$	S_k
$6^k \cdot 6 = (2^k + 4r)(2 + 4)$	*Multiply each side by a quantity equal to 6.*
$6^{k+1} = 2^{k+1} + 4(2^k) + 8r + 16r$	*Distributive Property*
$6^{k+1} - 2^{k+1} = 2^{k+1} + 4(2^k) + 8r + 16r - 2^{k+1}$	*Subtract 2^{k+1} from each side.*
$6^{k+1} - 2^{k+1} = 4 \cdot 2^k + 24r$	*Simplify.*
$6^{k+1} - 2^{k+1} = 4(2^k + 6r)$	*Factor.*

Let $t = 2^k + 6r$, an integer. Then $6^{k+1} - 2^{k+1} = 4t$.

We have shown that if S_k is valid, then S_{k+1} is also valid. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. Hence, $6^n - 2^n$ is divisible by 4 for all positive integers n.

Let's look at another proof that requires some creative thinking to complete.`

Example ④ **BUSINESS**   **Refer to the application at the beginning of the lesson. Use mathematical induction to prove that Emily can give change in \$2s and \$5s for amounts of \$4, \$5, \$6, …, \$n.**

Let n = the dollar amount for which a customer might need change.
Let a = the number of \$2 bills used to make the exchange.
Let b = the number of \$5 bills used to make the exchange.

$S \Rightarrow n = 2a + 5b$

Notice that the first possible case for n is not always 1.

1. First, verify that S_n is valid for $n = 4$. In other words, are there values for a and b such that $4 = 2a + 5b$?

   Yes. If $a = 2$ and $b = 0$, then $4 = 2(2) + 5(0)$, so S_n is valid for the first case.

2. Then, assume that S_n is valid for $n = k$ and prove that it is also valid for $n = k + 1$.

   $S_k \Rightarrow k = 2a + 5b$

   $S_{k+1} \Rightarrow k + 1 = 2a + 5b + 1$ *Add 1 to each side.*

   $\qquad\qquad = 2a + 5b + 6 - 5$ *$1 = 6 - 5$*

   $\qquad\qquad = 2(a + 3) + 5(b - 1)$ *$2a + 6 = 2(a + 3); 5b - 5 = 5(b - 1)$*

   Notice that this expression is true for all $a \geq 0$ and $b \geq 1$, since the expression $b - 1$ must be nonnegative. Therefore, we must also consider the case where $b = 0$.

   If $b = 0$, then $n = 2a + 5(0) = 2a$. Assuming S_n to be true for $n = k$, we must show that it is also true for $k + 1$.

   $S_k \Rightarrow k = 2a$

   $S_{k+1} \Rightarrow k + 1 = 2a + 1$

   $\qquad\qquad = 2(a - 2) + 4 + 1$ *$2a = 2(a - 2) + 4$*

   $\qquad\qquad = 2(a - 2) + 1 \cdot 5$ *$4 + 1 = 1 \cdot 5$*

   Notice that this expression is only true for all $a \geq 2$, since the expression $a - 2$ must be nonnegative. However, we have already considered the case where $a = 0$.

   Thus, it can be concluded that since the conjecture is true for $k = 4$, it also valid for $n = k + 1$. Therefore, Emily can make change for amounts of \$4, \$5, \$6, … \$n using only \$2 and \$5 bills.

◀ CONTENTS ▶

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why it is necessary to prove the $n + 1$ case in the process of mathematical induction.

2. **Describe** a method you might use to show that a conjecture is false.

3. Consider the series $3 + 5 + 7 + 9 + \cdots + 2n + 1$.
   a. **Write a conjecture** for the general formula S_n for the sum of the first n terms.
   b. Verify S_n for $n = 1$, 2, and 3.
   c. Write S_k and S_{k+1}.

4. **Restate** the conjecture that $8^n - 1$ is divisible by 7 for all positive integers n using the definition of divisibility.

5. *Math Journal*   The "domino effect" presented in this lesson is just one way to illustrate the principle behind mathematical induction. **Write** a paragraph describing another situation in real life that illustrates this principle.

Guided Practice

Use mathematical induction to prove that each proposition is valid for all positive integral values of *n*.
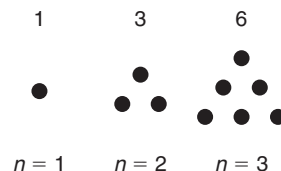
6. $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$

7. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$

8. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

9. $3^n - 1$ is divisible by 2.

10. **Number Theory**   The ancient Greeks were very interested in number patterns. Triangular numbers are numbers that can be represented by a triangular array of dots, with n dots on each side. The first three triangular numbers are 1, 3, and 6.

   a. Find the next five triangular numbers.
   b. Write a general formula for the nth term of this sequence.
   c. Prove that the sum of first n triangular numbers can be found using the formula $\dfrac{n(n + 1)(n + 2)}{6}$.

EXERCISES

Practice

For Exercises 11-19, use mathematical induction to prove that each proposition is valid for all positive integral values of *n*.

11. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

12. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

13. $-\dfrac{1}{2} - \dfrac{1}{4} - \dfrac{1}{8} - \cdots - \dfrac{1}{2^n} = \dfrac{1}{2^n} - 1$

14. $1 + 8 + 27 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$

15. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n - 1)(2n + 1)}{3}$

www.amc.glencoe.com/self_check_quiz

CONTENTS

16. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$

17. $7^n + 5$ is divisible by 6

18. $8^n - 1$ is divisible by 7

19. $5^n - 2^n$ is divisible by 3

20. Prove $S_n \Rightarrow a + (a + d) + (a + 2d) + \cdots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$.

21. Prove $S_n \Rightarrow \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

22. Prove $S_n \Rightarrow 2^{2n+1} + 3^{2n+1}$ is divisible by 5.

Applications and Problem Solving

Real World Application

23. Historical Proof In 1730, Abraham De Moivre proposed the following theorem for finding the power of a complex number written in polar form.

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

Prove that De Moivre's Theorem is valid for any positive integer n.

24. Number Patterns Consider the pattern of dots shown below.

a. Find the next figure in this pattern.

b. Write a sequence to represent the number of dots added to the previous figure to create the next figure in this pattern. The first term in your sequence should be 1.

c. Find the general form for the nth term of the sequence found in part **b**.

d. Determine a formula that will calculate the total number of dots for the nth figure in this dot pattern.

e. Prove that the formula found in part d is correct using mathematical induction.
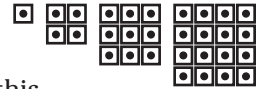
25. Critical Thinking Prove that $n^2 + 5n$ is divisible by 2 for all positive integral values of n.

26. Club Activites Melissa is the activities director for her school's science club and needs to coordinate a "get to know you" activity for the group's first meeting. The activity must last at least 40 minutes but no more than 60 minutes. Melissa has chosen an activity that will require each participant to interact with every other participant in attendance only once and estimates that if her directions are followed, each interaction should take approximately 30 seconds.

a. If n people participate in Melissa's activity, develop a formula to calculate the total number of such interactions that should take place.

b. Prove that the formula found in part a is valid for all positive integral values of n.

c. If 15 people participate, will Melissa's activity meet the guidelines provided to her? Explain.

27. **Critical Thinking**   Use mathematical induction to prove that the Binomial Theorem is valid for all positive integral values of n.

28. **Number Theory**   Consider the following statement: $0.99999 \cdots = 1$.
   a. Write $0.\overline{9}$ as an infinite series.
   b. Write an expression for the nth term of this series.
   c. Write a formula for the sum of the first n terms of this series.
   d. Prove that the formula you found in part **c** is valid for all positive integral values of n using mathematical induction.
   e. Use the formula found in part **c** to prove that $0.99999 \cdots = 1$. (*Hint:* Use what you know about the limits of infinite sequences.)
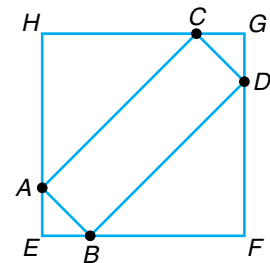
29. Find the first three iterates of the function $f(z) = 2z + i$ for $z_0 = 4 - i$.
   (Lesson 12-8)

30. Find d for the arithmetic sequence for which $a_1 = -6$ and $a_{29} = 64$.
   (Lesson 12-1)

31. Write the standard form of the equation $25x^2 + 4y^2 - 100x - 40y + 100 = 0$. Then identify the conic section this equation represents. *(Lesson 10-6)*

32. **Geology**   A *drumlin* is an elliptical streamlined hill whose shape can be expressed by the equation $r = \ell \cos k\theta$ for $-\frac{\pi}{2k} \le \theta \le \frac{\pi}{2k}$, where ℓ is the length of the drumlin and $k > 1$ is a parameter that is the ratio of the length to the width. Find the area in square centimeters, $A = \frac{\ell^2 \pi}{4k}$, of a drumlin modeled by the equation $r = 250 \cos 4\theta$. *(Lesson 9-3)*

glacier

kames   drumlins   esker
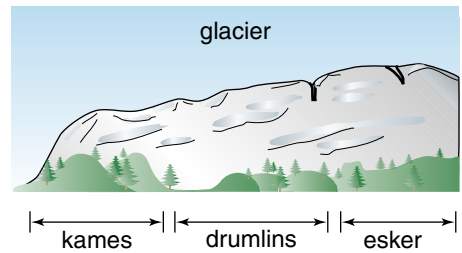
33. Write an equation of a sine function with amplitude $\frac{3}{4}$ and period π.
   (Lesson 6-4)

34. Find the values of x in the interval $0° \le x \le 360°$ for which $x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.
   (Lesson 5-5)

35. **SAT Practice**   The area of *EFGH* is 25 square units. Points A, B, C, and D are on the square. *ABCD* is a rectangle, but not a square. Calculate the perimeter of *ABCD* if the distance from E to A is 1 and the distance from E to B is 1.
   A 64 units
   B $10\sqrt{2}$ units
   C 10 units
   D 8 units
   E $5\sqrt{2}$ units

Extra Practice   See p. A50.

VOCABULARY

arithmetic mean (p. 760)
arithmetic sequence (p. 759)
arithmetic series (p. 761)
Binomial Theorem (p. 803)
common difference (p. 759)
common ratio (p. 766)
comparison test (p. 789)
convergent series (p. 786)
divergent series (p. 786)
escaping point (p. 818)
Euler's Formula (p. 809)
exponential series (p. 808)

Fibonacci sequence (p. 806)
fractal geometry (p. 817)
geometric mean (p. 768)
geometric sequence (p. 766)
geometric series (p. 769)
index of summation (p. 794)
infinite sequence (p. 774)
infinite series (p. 778)
limit (p. 774)
mathematical induction (p. 822)
n factorial (p. 796)

nth partial sum (p. 761)
orbit (p. 817)
Pascal's Triangle (p. 801)
prisoner point (p. 818)
ratio test (p. 787)
recursive formula (p. 760)
sequence (p. 759)
sigma notation (p. 794)
term (p. 759)
trigonometric series (p. 808)

UNDERSTANDING AND USING THE VOCABULARY

Choose the letter of the term that best matches each statement or phrase.

1. each succeeding term is formulated from one or more previous terms

2. ratio of successive terms in a geometric sequence

3. used to determine convergence

4. can have infinitely many terms

5. $e^{ix} = \cos \alpha + i \sin \alpha$

6. the terms between any two nonconsecutive terms of a geometric sequence

7. indicated sum of the terms of an arithmetic sequence

8. $n! = n(n - 1)(n - 2) \cdot \cdots \cdot 1$

9. used to demonstrate the validity of a conjecture based on the truth of a first case, the assumption of truth of a kth case, and the demonstration of truth for the $(k + 1)$th case

10. an infinite series with a sum or limit

a. term
b. mathematical induction
c. arithmetic series
d. recursive formula
e. n factorial
f. geometric mean
g. sigma notation
h. convergent
i. common ratio
j. infinite sequence
k. Euler's Formula
l. prisoner point
m. ratio test
n. limit

CONTENTS

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES	REVIEW EXERCISES

Lesson 12-1 Find the nth term and arithmetic means of an arithmetic sequence.

• Find the 35th term in the arithmetic sequence $-5, -1, 3, \ldots$.
Begin by finding the common difference d.
$d = -1 - (-5)$ or 4
Use the formula for the nth term.
$a_n = a_1 + (n - 1)d$
$a_{35} = -5 + (35 - 1)(4)$ or 131

11. Find the next four terms of the arithmetic sequence $3, 4.3, 5.6, \ldots$.

12. Find the 20th term of the arithmetic sequence for which $a_1 = 5$ and $d = -3$.

13. Form an arithemetic sequence that has three arithmetic means between 6 and -4.

Lesson 12-1 Find the sum of n terms of an arithmetic series.

• The sum S_n of the first n terms of an arithmetic series is given by
$S_n = \frac{n}{2}(a_1 + a_n)$.

14. What is the sum of the first 14 terms in the arithmetic series $-30 - 23 - 16 - \cdots$?

15. Find n for the arithmetic series for which $a_1 = 2$, $d = 1.4$, and $S_n = 250.2$.

Lesson 12-2 Find the nth term and geometric means of a geometric sequence.

• Find an approximation for the 12th term of the sequence $-8, 4, -2, 1, \ldots$.
First, find the common ratio.
$a_2 \div a_1 = 4 \div (-8)$ or -0.5
Use the formula for the nth term.
$a_{12} = -8(-0.5)^{12 - 1}$ $a_n = a_1 r^{n - 1}$
$= -8(-0.5)^{11}$ or about 0.004

16. Find the next three terms of the geometric sequence $49, 7, 1, \ldots$.

17. Find the 15th term of the geometric sequence for which $a_1 = 2.2$ and $r = 2$.

18. If $r = 0.2$ and $a_7 = 8$, what is the first term of the geometric sequence?

19. Write a geometric sequence that has three geometric means between 0.2 and 125.

Lesson 12-2 Find the sum of n terms of a geometric series.

• Find the sum of the first 12 terms of the geometric series $4 + 10 + 25 + 62.5 + \cdots$.
First find the common ratio.
$a_2 \div a_1 = 10 \div 4$ or 2.5
Now use the formula for the sum of a finite geometric series.
$S_n = \dfrac{a_1 - a_1 r^n}{1 - r}$
$S_{12} = \dfrac{4 - 4(2.5)^{12}}{1 - 2.5}$ $n = 12, a_1 = 4, r = 2.5$
$S_{12} \approx 158{,}943.05$ *Use a calculator.*

20. What is the sum of the first nine terms of the geometric series $1.2 - 2.4 + 4.8 - \cdots$?

21. Find the sum of the first eight terms of the geometric series $4 + 4\sqrt{2} + 8 + \cdots$.

CONTENTS

OBJECTIVES AND EXAMPLES

Lesson 12-3 Find the limit of the terms and the sum of an infinite geometric series.

- Find $\lim\limits_{n\to\infty} \dfrac{2n^2 + 5}{3n^2}$.

$$\lim_{n\to\infty} \frac{2n^2 + 5}{3n^2} = \lim_{n\to\infty}\left(\frac{2}{3} + \frac{5}{3n^2}\right)$$
$$= \lim_{n\to\infty} \frac{2}{3} + \lim_{n\to\infty} \frac{5}{3} \cdot \lim_{n\to\infty} \frac{1}{n^2}$$
$$= \frac{2}{3} + \frac{5}{3} \cdot 0$$

Thus, the limit is $\dfrac{2}{3}$.

REVIEW EXERCISES

Find each limit, or state that the limit does not exist and explain your reasoning.

22. $\lim\limits_{n\to\infty} \dfrac{3n}{4n + 1}$

23. $\lim\limits_{n\to\infty} \dfrac{6n - 3}{n}$

24. $\lim\limits_{n\to\infty} \dfrac{2^n\, n^3}{3n^3}$

25. $\lim\limits_{n\to\infty} \dfrac{4n^3 - 3n}{n^4 - 4n^3}$

26. Write $5.\overline{123}$ as a fraction.

27. Find the sum of the infinite series $1260 + 504 + 201.6 + 80.64 + \cdots$, or state that the sum does not exist and explain your reasoning.

Lesson 12-4 Determine whether a series is convergent or divergent.

- Use the ratio test to determine whether the series $3 + \dfrac{3^2}{2!} + \dfrac{3^3}{3!} + \dfrac{3^4}{4!}$ is convergent or divergent.

The nth term a_n of this series has a general form of $\dfrac{3^n}{n!}$ and $a_{n+1} = \dfrac{3^{n+1}}{(n+1)!}$. Find $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$.

$$r = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{(n+1)!}}{\dfrac{3^n}{n!}}$$

$$r = \lim_{n\to\infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right]$$

$r = \lim\limits_{n\to\infty} \dfrac{3}{n+1}$ or 0

Since $r < 0$, the series is convergent.

28. Use the ratio test to determine whether the series $\dfrac{1}{5} + \dfrac{2^2}{5^2} + \dfrac{3^2}{5^3} + \dfrac{4^2}{5^4} + \cdots$ is *convergent* or *divergent*.

29. Use the comparison test determine whether the series $\dfrac{6}{1} + \dfrac{7}{2} + \dfrac{8}{3} + \dfrac{9}{4} + \cdots$ is *convergent* or *divergent*.

30. Determine whether the series $2 + 1 + \dfrac{2}{3} + \dfrac{1}{2} + \dfrac{2}{5} + \dfrac{1}{3} + \dfrac{2}{7} + \cdots$ is *convergent* or *divergent*.

Lesson 12-5 Use sigma notation.

- Write $\sum\limits_{n=1}^{3} (n^2 - 1)$ in expanded form and then find the sum.

$$\sum_{n=1}^{3} (n^2 - 1) = (1^2 - 1) + (2^2 - 1) + (3^2 - 1)$$
$$= 0 + 3 + 8 \text{ or } 11$$

Write each expression in expanded form and then find the sum.

31. $\sum\limits_{a=5}^{9} (3a - 3)$

32. $\sum\limits_{k=1}^{\infty} (0.4)^k$

Express each series using sigma notation

33. $-1 + 1 + 3 + 5 + \cdots$

34. $2 + 5 + 10 + 17 + \cdots + 82$

CONTENTS

OBJECTIVES AND EXAMPLES

REVIEW EXERCISES

Lesson 12-6 Use the Binomial Theorem to expand binomials.

- Find the fourth term of $(2x - y)^6$.

$$(2x - y)^6 = \sum_{r=0}^{6} \frac{6!}{r!(6 - r)!} (2x)^{6 - r}(-y)^r$$

To find the fourth term, evaluate the general term for $r = 3$.

$$\frac{6!}{3!(6 - 3)!} (2x)^{6 - 3}(-y)^3$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!}(2x)^3(-y^3) \text{ or } -160x^3y^3$$

Use the Binomial Theorem to expand each binomial.

35. $(a - 4)^6$ **36.** $(2r + 3s)^4$

Find the designated term of each binomial expansion.

37. 5th term of $(x - 2)^{10}$

38. 3rd term of $(4m + 1)^8$

39. 8th term of $(x + 3y)^{10}$

40. 6th term of $(2c - d)^{12}$

Lesson 12-7 Use Euler's Formula to write the exponential form of a complex number.

- Write $\sqrt{3} - i$ in exponential form.

Write the polar form of $\sqrt{3} - i$.

$r = \sqrt{(\sqrt{3})^2 + (-1)^2}$ or 2, and

$\theta = \text{Arctan } \dfrac{-1}{\sqrt{3}}$ or $\dfrac{5\pi}{6}$

$$\sqrt{3} - i = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) = 2e^{i\frac{5\pi}{6}}$$

Write each expression or complex number in exponential form.

41. $2\left(\cos \dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4}\right)$

42. $4i$

43. $2 - 2i$

44. $3\sqrt{3} + 3i$

Lesson 12-8 Iterate functions using real and complex numbers.

- Find the first three iterates of the function $f(z) = 2z$ if the initial value is $3 - i$.

$z_0 = 3 - i$

$z_1 = 2(3 - i)$ or $6 - 2i$

$z_2 = 2(6 - 2i)$ or $12 - 4i$

$z_3 = 2(12 - 4i)$ or $24 - 8i$

Find the first four iterates of each function using the given initial value. If necessary, round your answers to the nearest hundredth.

45. $f(x) = 6 - 3x, x_0 = 2$

46. $f(x) = x^2 + 4, x_0 = -3$

Find the first three iterates of the function $f(z) = 0.5z + (4 - 2i)$ for each initial value.

47. $z_0 = 4i$ **48.** $z_0 = -8$

49. $z_0 = -4 + 6i$ **50.** $z_0 = 12 - 8i$

Lesson 12-9 Use mathematical induction to prove the validity of mathematical statements.

- Proof by mathematical induction:
  1. First, verify that the conjecture S_n is valid for the first possible case, usually $n = 1$.
  2. Then, assume that S_n is valid for $n = k$ and use this assumption to prove that it is also valid for $n = k + 1$.

Use mathematical induction to prove that each proposition is valid for all positive integral values of n.

51. $1 + 2 + 3 + \cdots + n = \dfrac{n(n + 1)}{2}$

52. $3 + 8 + 15 + \cdots + n(n - 2) = \dfrac{n(n + 1)(2n + 7)}{6}$

53. $9^n - 4^n$ is divisible by 5.

APPLICATIONS AND PROBLEM SOLVING

54. Physics If an object starting at rest falls in a vacuum near the surface of Earth, it will fall 16 feet during the first second, 48 feet during the next second, 80 feet during the third second, and so on. *(Lesson 12-1)*

   a. How far will the object fall during the twelfth second?

   **b.** How far will the object have fallen after twelve seconds?

55. Budgets A major corporation plans to cut the budget on one of its projects by 3 percent each year. If the current budget for the project to be cut is $160 million, what will the budget for that project be in 10 years? *(Lesson 12-2)*

56. Geometry If the midpoints of the sides of an equilateral triangle are joined by straight lines, the new figure will also be an equilateral triangle. *(Lesson 12-3)*

   a. If the original triangle has a perimeter of 6 units, find the perimeter of the new triangle.
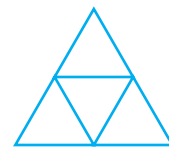
   b. If this process is continued to form a sequence of "nested" triangles, what will be the sum of the perimeters of all the triangles?
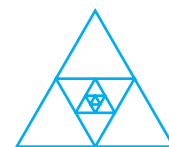
ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. A sequence has a common difference of 3.

   a. Is this sequence arithmetic or geometric? Explain.

   b. Form a sequence that has this common difference. Write a recursive formula for your sequence.

2. Write a general expression for an infinite sequence that has no limit. Explain your reasoning.

PORTFOLIO
Explain the difference between a convergent and a divergent series. Give an example of each type of series and show why it is that type of series.

Additional Assessment See p. A67 for Chapter 12 practice test.
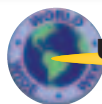
Unit 4 *inter*NET Project

THE UNITED STATES CENSUS BUREAU

That's a lot of people!

- Use the Internet to find the population of the United States from at least 1900 through 2000. Write a sequence using the population for each ten-year interval, for example, 1900, 1910, and so on.

- Write a formula for an arithmetic sequence that provides a reasonable model for the population sequence.

- Write a formula for a geometric sequence that provides a reasonable model for the population sequence.

- Use your models to predict the U.S. population for the year 2050. Write a one-page paper comparing the arithmetic and geometric sequences you used to model the population data. Discuss which formula you think best models the data.

CONTENTS

Percent Problems

A few common words and phrases used in percent problems, along with their translations into mathematical expressions, are listed below.

what ➡ x (variable)	What percent of A is B? ➡ $\dfrac{x}{100} \cdot A = B$
of ➡ × (multiply)	What is A percent more than B? ➡ $x = B + \dfrac{A}{100} \cdot B$
is ➡ = (equals)	What is A percent less than B? ➡ $x = B - \dfrac{A}{100} \cdot B$

percent of change (increase or decrease) ➡ $\dfrac{\text{amount of change}}{\text{original amount}} \times 100$

THE PRINCETON REVIEW

TEST-TAKING TIP
With common percents, like 10%, 25%, or 50%, it is faster to use the fraction equivalents and mental math than a calculator.

ACT EXAMPLE

1. If c is positive, what percent of $3c$ is 9?

A $\dfrac{c}{100}\%$ **B** $\dfrac{300}{c}\%$ **C** $\dfrac{9}{c}\%$ **D** 3% **E** $\dfrac{c}{3}\%$

HINT Use variables just as you would use numbers.

Solution Start by translating the question into an equation.

What percent of $3c$ is 9?

$$\dfrac{x}{100} \cdot 3c = 9$$

Now solve the equation for x, not c.

$\dfrac{x}{100} = \dfrac{9}{3c}$ *Divide each side by 3c.*

$\dfrac{x}{100} = \dfrac{3}{c}$ *Simplify.*

$x = \dfrac{300}{c}$ *Solve for x.*

The answer is choice **B**.

Alternate Solution "Plug in" 3 for c. The question becomes "what percent of 9 is 9?" The answer is 100%, so check each expression choice to see if it is equal to 100% when $c = 3$.

Choice A: $\dfrac{c}{100}\% = \dfrac{3}{100}\%$

Choice B: $\dfrac{300}{c}\% = 100\%$

Choice **B** is correct.

SAT EXAMPLE

2. An electronics store offers a 25% discount on all televisions during a sale week. How much must a customer pay for a television marked at $240?

A $60   **B** $300   **C** $230.40

D $180   **E** $215

HINT A discount is a decrease in the price of an item. So, the question asked is "What is 25% less than 240?"

Solution Start by translating this question into an equation.

What is 25% less than 240? ➡ $x = 240 - \dfrac{25}{100} \cdot 240$

Now simplify the right-hand side of the equation.

$$x = 240 - \dfrac{1}{4} \cdot 240 \text{ or } 180$$

Choice **D** is correct.

Alternate Solution If there is a 25% discount, a customer will pay $(100 - 25)\%$ or 75% of the marked price.

What is 75% of the marked price? ➡ $x = 0.75 \cdot 240 \text{ or } 180$
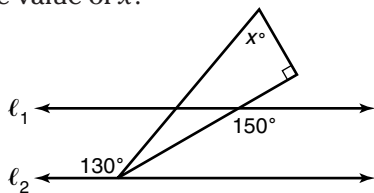
The answer is choice **D**.

CONTENTS

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. Shanika has a collection of 80 tapes. If 40% of her records are jazz tapes and the rest are blues tapes, how many blues tapes does she have?

  A 32 **B** 40 **C** 42 **D** 48 **E** 50

2. If ℓ_1 is parallel to ℓ_2 in the figure below, what is the value of x?

  A 20 **B** 50 **C** 70 **D** 80 **E** 90

3. There are k gallons of gasoline available to fill a tank. After d gallons have been pumped, then, in terms of k and d, what percent of the gasoline has been pumped?

  A $\frac{100d}{k}\%$ **B** $\frac{k}{100d}\%$ **C** $\frac{100k}{d}\%$

  D $\frac{k}{100(k-d)}\%$ **E** $\frac{100(k-d)}{k}\%$

4. In 1985, Andrei had a collection of 48 baseball caps. Since then he has given away 13 caps, purchased 17 new caps, and traded 6 of his caps to Pierre for 8 of Pierre's caps. Since 1985, what has been the net percent increase in Andrei's collection?

  A 6% **B** $12\frac{1}{2}\%$ **C** $16\frac{2}{3}\%$

  D 25% **E** $28\frac{1}{2}\%$

5. In the figure below, $AB = AC$ and AD is a line segment. What is the value of $x - y$?
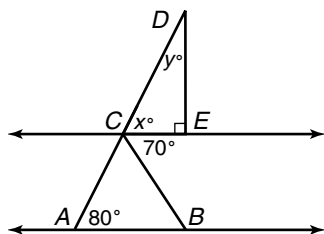
Note: Figure is NOT drawn to scale.

  A 10 **B** 20 **C** 30 **D** 70 **E** 90

6. At the beginning of 2000, the population of Rockville was 204,000, and the population of Springfield was 216,000. If the population of each city increased by exactly 20% in 2000, how many more people lived in Springfield than in Rockville at the end of 2000?

  A 9,600 **B** 10,000 **C** 12,000

  D 14,400 **E** 20,000

7. In the figure, the slope of $\overline{AC}$ is $-\frac{1}{6}$, and $m\angle C = 30°$. What is the length of $\overline{BC}$?
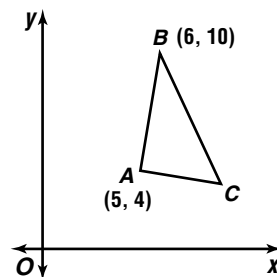
  A $\sqrt{37}$
  B $\sqrt{111}$
  C 2
  D $2\sqrt{37}$
  E It cannot be determined from the information given.

8. If $x + 6 > 0$ and $1 - 2x > -1$, then x could equal each of the following EXCEPT ?

  A -6 **B** -4 **C** -2 **D** 0 **E** $\frac{1}{2}$

9. The percent increase from 99 to 100 is which of the following?
  A greater than 1
  B 1
  C less than 1, but more than $\frac{1}{2}$
  D less than $\frac{1}{2}$, but more than 0
  E 0

10. Grid-In One fifth of the cars in a parking lot are blue, and $\frac{1}{2}$ of the blue cars are convertibles. If $\frac{1}{4}$ of the convertibles in the parking lot are blue, then what percent of the cars in the lot are neither blue nor convertibles?

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