UNIT 4

Discrete Mathematics

Discrete mathematics is a branch of mathematics that deals with finite or discontinuous quantities. Usually discrete mathematics is defined in terms of its key topics. These include graphs, certain functions, logic, combinatorics, sequences, iteration, algorithms, recursion, matrices, and induction. Some of these topics have already been introduced in this book. For example, linear functions are continuous, while step functions are discrete. As you work through Unit 4, you will construct models, discover and use algorithms, and examine exciting new concepts as you solve real-world problems.

Chapter 12 **Sequences and Series**
Chapter 13 **Combinatorics and Probability**
Chapter 14 **Statistics and Data Analysis**

Unit 4 interNET Projects

THE UNITED STATES CENSUS BUREAU

Did you know that the very first United States Census was conducted in 1790? The United States Census Bureau completed this task in less than nine months. This is remarkable considering they had to do so without benefit of cars, telephones, or computers! As the U.S. population grew, it took more and more time to complete the census. The 1880 census took seven years to finish. Luckily, advancements in technology have helped the people employed by the Census Bureau analyze the data and prepare reports in a reasonable amount of time. At the end of each chapter in Unit 4, you will be given tasks to explore the data collected by the Census Bureau.

CHAPTER 12 (page 833)

That's a lot of people! In addition to determining how many people are in the United States every ten years, the Census Bureau also attempts to estimate the population at any particular time and in the future. On their web site, you can see estimates of the U. S. and world population for the current day. How does the Census Bureau estimate population?
Math Connection: Use the Internet to find population data. Model the population of the U.S. by using an arithmetic and then a geometric sequence. Predict the population for a future date using your models.

CHAPTER 13 (page 885)

Radically random! During a census year, the Census Bureau collects many type of data about people. This information includes age, ethnic background, and income, to name just a few. What types of data about the people of the United States can be found using the Internet?
Math Connection: Use data from the Internet to find the probability that a randomly-selected person in the U.S. belongs to a particular age group.

CHAPTER 14 (page 937)

More and more models! Did you know that in 1999 a person was born in the United States about every eight seconds? In that same year, about one person died every 15 seconds, and one person migrated every 20 seconds. Birth, deaths, and other factors directly affect the population. In Chapter 12, you modeled the U.S. population using sequences. What other types of population models could you use to predict population growth?
Math Connection: Use data from the Internet to write and graph several functions representing the U.S. population growth. Predict the population for a future data using your models.

interNET CONNECTION

For more information on the Unit Project, visit:
www.amc.glencoe.com

Chapter 12

Sequences and Series

CHAPTER OBJECTIVES

- **Identify and find *n*th terms of arithmetic, geometric, and infinite sequences.** *(Lessons 12-1, 12-2)*
- **Find sums of arithmetic, geometric, and infinite series.** *(Lessons 12-1, 12-2, 12-3)*
- **Determine whether a series is convergent or divergent.** *(Lesson 12-4)*
- **Use sigma notation.** *(Lesson 12-5)*
- **Use the Binomial Theorem to expand binomials.** *(Lesson 12-6)*
- **Evaluate expressions using exponential, trigonometric, and iterative series.** *(Lessons 12-7, 12-8)*
- **Use mathematical induction to prove the validity of mathematical statements.** *(Lesson 12-9)*

12-1

Arithmetic Sequences and Series

OBJECTIVES
- Find the nth term and arithmetic means of an arithmetic sequence.
- Find the sum of n terms of an arithmetic series.

Real World Application

REAL ESTATE Ofelia Gonzales sells houses in a new development. She makes a commission of $3750 on the sale of her first house. To encourage aggressive selling, Ms. Gonzales' employer promises a $500 increase in commission for each additional house sold. Thus, on the sale of her next house, she will earn $4250 commission. How many houses will Ms. Gonzales have to sell for her total commission in one year to be at least $65,000? *This problem will be solved in Example 6.*

The set of numbers representing the amount of money earned for each house sold is an example of a **sequence.** A sequence is a function whose domain is the set of natural numbers. The **terms** of a sequence are the range elements of the function. The first term of a sequence is denoted a_1, the second term is a_2, and so on up to the nth term a_n.

Look Back
Refer to Lesson 4-1 for more about natural numbers.

Symbol	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
Term	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-1\frac{1}{2}$

The sequence given in the table above is an example of an **arithmetic sequence.** The difference between successive terms of an arithmetic sequence is a constant called the **common difference,** denoted d. In the example above, $d = \frac{1}{2}$.

Arithmetic Sequence

An arithmetic sequence is a sequence in which each term after the first, a_1, is equal to the sum of the preceding term and the common difference, d. The terms of the sequence can be represented as follows.

$$a_1,\ a_1 + d,\ a_1 + 2d,\ \ldots$$

To find the next term in an arithmetic sequence, first find the common difference by subtracting any term from its succeeding term. Then add the common difference to the last term to find the next term in the sequence.

Example 1 **Find the next four terms in the arithmetic sequence $-5, -2, 1, \ldots$**

First, find the common difference.

$a_2 - a_1 = -2 - (-5)$ or 3
$a_3 - a_2 = 1 - (-2)$ or 3

Find the difference between pairs of consecutive terms to verify the common difference.

The common difference is 3.

Add 3 to the third term to get the fourth term, and so on.

$a_4 = 1 + 3$ or 4 $\quad a_5 = 4 + 3$ or 7 $\quad a_6 = 7 + 3$ or 10 $\quad a_7 = 10 + 3$ or 13

The next four terms are 4, 7, 10, and 13.

By definition, the nth term is also equal to $a_{n-1} + d$, where a_{n-1} is the $(n-1)$th term. That is, $a_n = a_{n-1} + d$. This type of formula is called a **recursive formula.** This means that each succeeding term is formulated from one or more previous terms.

The nth term of an arithmetic sequence can also be found when only the first term and the common difference are known. Consider an arithmetic sequence in which $a = -3.7$ and $d = 2.9$. Notice the pattern in the way the terms are formed.

first term	a_1	a	-3.7
second term	a_2	$a + d$	$-3.7 + 1(2.9) = -0.8$
third term	a_3	$a + 2d$	$-3.7 + 2(2.9) = 2.1$
fourth term	a_4	$a + 3d$	$-3.7 + 3(2.9) = 5.0$
fifth term	a_5	$a + 4d$	$-3.7 + 4(2.9) = 7.9$
⋮	⋮	⋮	⋮
nth term	a_n	$a + (n-1)d$	$-3.7 + (n-1)2.9$

The *n*th Term of an Arithmetic Sequence

The nth term of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n-1)d$.

Notice that the preceding formula has four variables: a_n, a_1, n, and d. If any three of these are known, the fourth can be found.

Examples

2 Find the 47th term in the arithmetic sequence −4, −1, 2, 5, ….

First, find the common difference.

$a_2 - a_1 = -1 - (-4)$ or 3 $\quad a_3 - a_2 = 2 - (-1)$ or 3 $\quad a_4 - a_3 = 5 - 2$ or 3

The common difference is 3.

Then use the formula for the nth term of an arithmetic sequence.

$$a_n = a_1 + (n-1)d$$
$$a_{47} = -4 + (47-1)3 \quad \textit{n = 47, } a_1 = -4\textit{, and d = 3}$$
$$a_{47} = 134$$

3 Find the first term in the arithmetic sequence for which $a_{19} = 42$ and $d = -\frac{2}{3}$.

$$a_n = a_1 + (n-1)d$$
$$a_{19} = a_1 + (19-1)\left(-\frac{2}{3}\right) \quad n = 19 \text{ and } d = -\frac{2}{3}$$
$$42 = a_1 + (-12) \quad a_{19} = 42$$
$$a_1 = 54$$

Sometimes you may know two terms of an arithmetic sequence that are not in consecutive order. The terms between any two nonconsecutive terms of an arithmetic sequence are called **arithmetic means.** In the sequence below, 38 and 49 are the arithmetic means between 27 and 60.

5, 16, 27, 38, 49, 60

Example 4 **Write an arithmetic sequence that has five arithmetic means between 4.9 and 2.5.**

The sequence will have the form 4.9, __?__, __?__, __?__, __?__, __?__, 2.5. Note that 2.5 is the 7th term of the sequence or a_7.

First, find the common difference, using $n = 7$, $a_7 = 2.5$, and $a_1 = 4.9$

$$a_n = a_1 + (n - 1)d$$
$$2.5 = 4.9 + (7 - 1)d$$
$$2.5 = 4.9 + 6d$$
$$d = -0.4$$

Then determine the arithmetic means.

$$a_2 = 4.9 + (-0.4) \text{ or } 4.5$$
$$a_3 = 4.5 + (-0.4) \text{ or } 4.1$$
$$a_4 = 4.1 + (-0.4) \text{ or } 3.7$$
$$a_5 = 3.7 + (-0.4) \text{ or } 3.3$$
$$a_6 = 3.3 + (-0.4) \text{ or } 2.9$$

The sequence is 4.9, 4.5, 4.1, 3.7, 3.3, 2.9, 2.5.

An indicated sum is 1 + 2 + 3 + 4. The sum 1 + 2 + 3 + 4 is 10.

An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence. The lists below show some examples of arithmetic sequences and their corresponding arithmetic series.

Arithmetic Sequence	Arithmetic Series
$-9, -3, 3, 9, 15$	$-9 + (-3) + 3 + 9 + 15$
$3, \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}$	$3 + \frac{5}{2} + 2 + \frac{3}{2} + 1 + \frac{1}{2}$
$a_1, a_2, a_3, a_4, \ldots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

The symbol S_n, called the ***n*th partial sum**, is used to represent the sum of the first n terms of a series. To develop a formula for S_n for a finite arithmetic series, a series can be written in two ways and added term by term, as shown below. The second equation for S_n given below is obtained by reversing the order of the terms in the series.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n \\ + S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1 \\ \hline 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) \\ 2S_n &= n(a_1 + a_n) \end{aligned}$$

There are n terms in the series, all of which are $(a_1 + a_n)$.

Therefore, $S_n = \frac{n}{2}(a_1 + a_n)$.

Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

Example 5 **Find the sum of the first 60 terms in the arithmetic series 9 + 14 + 19 + ⋯ + 304.**

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{60} = \frac{60}{2}(9 + 304) \quad n = 60, a_1 = 9, a_{60} = 304$$
$$= 9390$$

When the value of the last term, a_n, is not known, you can still determine the sum of the series. Using the formula for the nth term of an arithmetic sequence, you can derive another formula for the sum of a finite arithmetic series.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{n}{2}[a_1 + (a_1 + (n - 1)d)] \quad a_n = a_1 + (n - 1)d$$

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

Example 6 Real World Application

REAL ESTATE **Refer to the application at the beginning of the lesson. How many houses will Ms. Gonzales have to sell for her total commission in one year to be at least $65,000?**

Let S_n = the amount of her desired commission, $65,000.
Let a_1 = the first commission, $3750.
In this example, $d = 500$.

We want to find n, the number of houses that Ms. Gonzales has to sell to have a total commission greater than or equal to $65,000.

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

$$65{,}000 = \frac{n}{2}[2(3750) + (n - 1)(500)] \quad S_n = 65{,}000,\ a_1 = 3750$$

$$130{,}000 = n(7500 + 500n - 500) \quad \text{Multiply each side by 2.}$$

$$130{,}000 = 7000n + 500n^2 \quad \text{Simplify.}$$

$$0 = 500n^2 + 7000n - 130{,}000$$

$$0 = 5n^2 + 70n - 1300 \quad \text{Divide each side by 100.}$$

$$n = \frac{-70 \pm \sqrt{70^2 - 4(5)(-1300)}}{2(5)} \quad \text{Use the Quadratic Formula.}$$

$$n = \frac{-70 \pm \sqrt{30{,}900}}{10}$$

$$n \approx 10.58 \text{ and } -24.58 \quad -24.58 \text{ is not a possible answer.}$$

Ms. Gonzales must sell 11 or more houses for her total commission to be at least $65,000.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** the first five terms of the sequence defined by $a_n = 6 - 4n$. Is this an arithmetic sequence? Explain.
2. Consider the arithmetic sequence defined by $a_n = \frac{5 - 2n}{2}$.
   a. **Graph** the first five terms of the sequence. Let n be the x-coordinate and a_n be the y-coordinate, and connect the points.
   b. **Describe** the graph found in part **a**.
   c. **Find** the common difference of the sequence and determine its relationship to the graph found in part **a**.

3. Refer to Example 6.

a. **Explain** why -24.58 is *not* a possible answer.

b. **Determine** how much money Ms. Gonzales will make if she sells 10 houses.

4. **Describe** the common difference for an arithmetic sequence in which the terms are decreasing.

5. **You Decide** Ms. Brooks defined two sequences, $a_n = (-1)^n$ and $b_n = (-2)^n$, for her class. She asked the class to determine if they were arithmetic sequences. Latonya said the second was an arithmetic sequence and that the first was not. Diana thought the reverse was true. Who is correct? Explain.

Guided Practice

Find the next four terms in each arithmetic sequence.

6. 6, 11, 16, …

7. $-15, -7, 1, \ldots$

8. $a - 6, a - 2, a + 2, \ldots$

For Exercises 9-15, assume that each sequence or series is arithmetic.

9. Find the 17th term in the sequence for which $a_1 = 10$ and $d = -3$.

10. Find n for the sequence for which $a_n = 37$, $a_1 = -13$, and $d = 5$.

11. What is the first term in the sequence for which $d = -2$ and $a_7 = 3$?

12. Find d for the sequence for which $a_1 = 100$ and $a_{12} = 34$.

13. Write a sequence that has two arithmetic means between 9 and 24.

14. What is the sum of the first 35 terms in the series $7 + 9 + 11 + \cdots$?

15. Find n for a series for which $a_1 = 30$, $d = -4$, and $S_n = -210$.

16. **Theater Design** The right side of the orchestra section of the Nederlander Theater in New York City has 19 rows, and the last row has 27 seats. The numbers of seats in each row increase by 1 as you move toward the back of the section. How many seats are in this section of the theater?

EXERCISES

Practice

Find the next four terms in each arithmetic sequence.

17. $5, -1, -7, \ldots$

18. $-18, -7, 4, \ldots$

19. 3, 4.5, 6, …

20. 5.6, 3.8, 2, …

21. $b, b + 4, b + 8, \ldots$

22. $-x, 0, x, \ldots$

23. $5n, -n, -7n, \ldots$

24. $5 + k, 5, 5 - k, \ldots$

25. $2a - 5, 2a + 2, 2a + 9, \ldots$

26. Determine the common difference and find the next three terms of the arithmetic sequence $3 + \sqrt{7}, 5, 7 - \sqrt{7}, \ldots$.

For Exercises 27-34, assume that each sequence or series is arithmetic.

27. Find the 25th term in the sequence for which $a_1 = 8$ and $d = 3$.

28. Find the 18th term in the sequence for which $a_1 = 1.4$ and $d = 0.5$.

29. Find n for the sequence for which $a_n = -41$, $a_1 = 19$, and $d = -5$.

30. Find n for the sequence for which $a_n = 138$, $a_1 = -2$, and $d = 7$.

31. What is the first term in the sequence for which $d = -3$, and $a_{15} = 38$?

32. What is the first term in the sequence for which $d = \frac{1}{3}$ and $a_7 = 10\frac{2}{3}$?

33. Find d for the sequence in which $a_1 = 6$ and $a_{14} = 58$.

34. Find d for the sequence in which $a_1 = 8$ and $a_{11} = 26$.

For Exercises 35-49, assume that each sequence or series is arithmetic.

35. What is the eighth term in the sequence $-4 + \sqrt{5}, -1 + \sqrt{5}, 2 + \sqrt{5}, \ldots$?

36. What is the twelfth term in the sequence $5 - \boldsymbol{i}, 6, 7 + \boldsymbol{i}, \ldots$?

37. Find the 33rd term in the sequence 12.2, 10.5, 8.8, ….

38. Find the 79th term in the sequence $-7, -4, -1, \ldots$.

39. Write a sequence that has one arithmetic mean between 12 and 21.

40. Write a sequence that has two arithmetic means between -5 and 4.

41. Write a sequence that has two arithmetic means between $\sqrt{3}$ and 12.

42. Write a sequence that has three arithmetic means between 2 and 5.

43. Find the sum of the first 11 terms in the series $\frac{3}{2} + 1 + \frac{1}{2} + \ldots$.

44. Find the sum of the first 100 terms in the series $-5 - 4.8 - 4.6 - \ldots$.

45. Find the sum of the first 26 terms in the series $-19 - 13 - 7 - \ldots$.

46. Find n for a series for which $a_1 = -7$, $d = 1.5$, and $S_n = -14$.

47. Find n for a series for which $a_1 = -3$, $d = 2.5$, and $S_n = 31.5$.

48. Write an expression for the nth term of the sequence 5, 7, 9, ….

49. Write an expression for the nth term of the sequence $6, -2, -10, \ldots$.

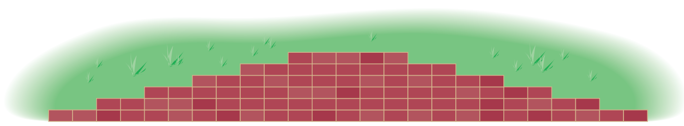
Applications and Problem Solving

Real World Application

50. Keyboarding Antonio has found that he can input statistical data into his computer at the rate of 2 data items faster each half hour he works. One Monday, he starts work at 9:00 A.M., inputting at a rate of 3 data items per minute. At what rate will Antonio be inputting data into the computer by lunchtime (noon)?

51. Critical Thinking Show that if x, y, z, and w are the first four terms of an arithmetic sequence, then $x + w - y = z$.

52. Construction The Arroyos are planning to build a brick patio that approximates the shape of a trapezoid. The shorter base of the trapezoid needs to start with a row of 5 bricks, and each row must increase by 2 bricks on each side until there are 25 rows. How many bricks do the Arroyos need to buy?



53. Critical Thinking The measures of the angles of a convex polygon form an arithmetic sequence. The least measurement in the sequence is 85°. The greatest measurement is 215°. Find the number of sides in this polygon.

54. Geometry The sum of the interior angles of a triangle is 180°.

a. What are the sums of the interior angles of polygons with 4, 5, 6, and 7 sides?

b. Show that these sums (beginning with the triangle) form an arithmetic sequence.

c. Find the sum of the interior angles of a 35-sided polygon.

55. Critical Thinking Consider the sequence of odd natural numbers.

a. What is S_5?

b. What is S_{10}?

c. Make a conjecture as to the pattern that emerges concerning the sum. Write an algebraic proof verifying your conjecture.

56. Sports At the 1998 Winter X-Games held in Crested Butte, Colorado, Jennie Waara, from Sweden, won the women's snowboarding slope-style competition. Suppose that in one of the qualifying races, Ms. Waara traveled 5 feet in the first second, and the distance she traveled increased by 7 feet each subsequent second. If Ms. Waara reached the finish line in 15 seconds, how far did she travel?

57. Entertainment A radio station advertises a contest with ten cash prizes totaling \$5510. There is to be a \$100 difference between each successive prize. Find the amounts of the least and greatest prizes the radio station will award.

58. Critical Thinking Some sequences involve a pattern but are not arithmetic. Find the sum of the first 72 terms in the sequence 6, 8, 2, …, where $a_n = a_{n-1} - a_{n-2}$.

Mixed Review

59. Personal Finance If Parker Hamilton invests \$100 at 7% compounded continuously, how much will he have at the end of 15 years? *(Lesson 11-3)*

60. Find the coordinates of the center, foci, and vertices of the ellipse whose equation is $4x^2 + 25y^2 + 250y + 525 = 0$. Then graph the equation. *(Lesson 10-3)*

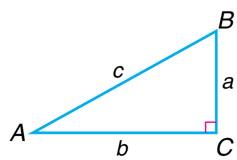
61. Find $6\left(\cos \frac{5\pi}{8} + \boldsymbol{i} \sin \frac{5\pi}{8}\right) \div 12\left(\cos \frac{\pi}{2} + \boldsymbol{i} \sin \frac{\pi}{2}\right)$. Then express the quotient in rectangular form. *(Lesson 9-7)*

62. Find the inner product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if $\vec{\mathbf{u}} = \langle 2, -1, 3 \rangle$ and $\vec{\mathbf{v}} = \langle 5, 3, 0 \rangle$. *(Lesson 8-4)*

63. Write the standard form of the equation of a line for which the length of the normal is 5 units and the normal makes an angle of 30° with the positive x-axis. *(Lesson 7-6)*

64. Graph $y = \sec 2\theta - 3$. *(Lesson 6-7)*

65. Solve triangle ABC if $B = 19°32'$ and $c = 4.5$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-5)*



66. Find the discriminant of $4p^2 - 3p + 2 = 0$. Describe the nature of its roots. *(Lesson 4-2)*

67. Determine the slant asymptote of $f(x) = \frac{x^2 - 4x + 2}{x - 3}$. *(Lesson 3-7)*

68. Triangle ABC is represented by the matrix $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 3 & -4 \end{bmatrix}$. Find the image of the triangle after a rotation of 270° counterclockwise about the origin. *(Lesson 2-4)*

69. SAT/ACT Practice If $a - 4b = 15$ and $4a - b = 15$, then $a - b = ?$

A 3 **B** 4 **C** 6 **D** 15 **E** 30

Extra Practice See p. A48.

12-2 Geometric Sequences and Series

OBJECTIVES
- Find the *n*th term and geometric means of a geometric sequence.
- Find the sum of *n* terms of a geometric series.

Real World Application

ACCOUNTING Bertha Blackwell is an accountant for a small company. On January 1, 1996, the company purchased $50,000 worth of office copiers. Since this equipment is a company asset, Ms. Blackwell needs to determine how much the copiers are presently worth. She estimates that copiers depreciate at a rate of 45% per year. What value should Ms. Blackwell assign the copiers on her 2001 year-end accounting report? *This problem will be solved in Example 3.*

The following sequence is an example of a **geometric sequence.**

10, 2, 0.4, 0.08, 0.016, … *Can you find the next term?*

The ratio of successive terms in a geometric sequence is a constant called the **common ratio,** denoted *r*.

Geometric Sequence

A geometric sequence is a sequence in which each term after the first, a_1, is the product of the preceding term and the common ratio, r. The terms of the sequence can be represented as follows, where a_1 is nonzero and r is not equal to 1 or 0.

$$a_1,\ a_1r,\ a_1r^2,\ \ldots.$$

You can find the next term in a geometric sequence as follows.

- First divide any term by the preceding term to find the common ratio.
- Then multiply the last term by the common ratio to find the next term in the sequence.

Example 1 **Determine the common ratio and find the next three terms in each sequence.**

a. $1, -\frac{1}{2}, \frac{1}{4}, \ldots$

First, find the common ratio.

$a_2 \div a_1 = -\frac{1}{2} \div 1$ or $-\frac{1}{2}$ $\qquad$ $a_3 \div a_2 = \frac{1}{4} \div \left(-\frac{1}{2}\right)$ or $-\frac{1}{2}$

The common ratio is $-\frac{1}{2}$.

Multiply the third term by $-\frac{1}{2}$ to get the fourth term, and so on.

$a_4 = \frac{1}{4} \cdot \left(-\frac{1}{2}\right)$ or $-\frac{1}{8}$ $\qquad$ $a_5 = -\frac{1}{8} \cdot \left(-\frac{1}{2}\right)$ or $\frac{1}{16}$ $\qquad$ $a_6 = \frac{1}{16} \cdot \left(-\frac{1}{2}\right)$ or $-\frac{1}{32}$

The next three terms are $-\frac{1}{8}, \frac{1}{16}$, and $-\frac{1}{32}$.

b. $r - 1, -3r + 3, 9r - 9, \ldots$

First, find the common ratio.

$a_2 \div a_1 = \frac{-3r + 3}{r - 1}$

$a_2 \div a_1 = \frac{-3(r - 1)}{r - 1}$ *Factor.*

$a_2 \div a_1 = -3$ *Simplify.*

$a_3 \div a_2 = \frac{9r - 9}{-3r + 3}$

$a_3 \div a_2 = \frac{9(r - 1)}{-3(r - 1)}$ *Factor.*

$a_3 \div a_2 = -3$ *Simplify.*

The common ratio is -3.

Multiply the third term by -3 to get the fourth term, and so on.

$a_4 = -3(9r - 9)$ or $-27r + 27$

$a_5 = -3(-27r + 27)$ or $81r - 81$

$a_6 = -3(81r - 81)$ or $-243r + 243$

The next three terms are $-27r + 27$, $81r - 81$, and $-243r + 243$.

As with arithmetic sequences, geometric sequences can also be defined recursively. By definition, the nth term is also equal to $a_{n-1}r$, where a_{n-1} is the $(n - 1)$th term. That is, $a_n = a_{n-1}r$.

Since successive terms of a geometric sequence can be expressed as the product of the common ratio and the previous term, it follows that each term can be expressed as the product of a_1 and a power of r. The terms of a geometric sequence for which $a_1 = -5$ and $r = 7$ can be represented as follows.

first term	a_1	a_1	-5
second term	a_2	a_1r	$-5 \cdot 7^1 = -35$
third term	a_3	a_1r^2	$-5 \cdot 7^2 = -245$
fourth term	a_4	a_1r^3	$-5 \cdot 7^3 = -1715$
fifth term	a_5	a_1r^4	$-5 \cdot 7^4 = -12{,}005$
⋮	⋮	⋮	⋮
nth term	a_n	ar^{n-1}	$-5 \cdot 7^{n-1}$

The nth Term of a Geometric Sequence

The nth term of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1r^{n-1}$.

Example 2 **Find an approximation for the 23rd term in the sequence 256, −179.2, 125.44, …**

First, find the common ratio.

$a_2 \div a_1 = -179.2 \div 256$ or -0.7

$a_3 \div a_2 = 125.44 \div (-179.2)$ or -0.7

The common ratio is -0.7.

Then, use the formula for the nth term of a geometric sequence.

$a_n = a_1r^{n-1}$

$a_{23} = 256(-0.7)^{23-1}$ *$n = 23$, $a_1 = 256$, $r = -0.7$*

$a_{23} \approx 0.1000914188$ *Use a calculator.*

The 23rd term is about 0.1.

Geometric sequences can represent growth or decay.

- For a common ratio greater than 1, a sequence may model growth. Applications include compound interest, appreciation of property, and population growth.
- For a positive common ratio less than 1, a sequence may model decay. Applications include some radioactive behavior and depreciation.

Example 3 Real World Application

ACCOUNTING **Refer to the application at the beginning of the lesson. Compute the value of the copiers at the end of the year 2001.**

Since the copiers were purchased at the beginning of the first year, the original purchase price of the copiers represents a_1. If the copiers depreciate at a rate of 45% per year, then they retain 100 − 45 or 55% of their value each year.

Use the formula for the nth term of a geometric sequence to find the value of the copiers six years later or a_7.

$a_n = a_1 r^{n-1}$

$a_7 = 50{,}000 \cdot (0.55)^{7-1}$ *$a_1 = 50{,}000$, $r = 0.55$, $n = 7$*

$a_7 \approx 1384.032031$ *Use a calculator.*

Ms. Blackwell should list the value of the copiers on her report as $1384.03.

The terms between any two nonconsecutive terms of a geometric sequence are called **geometric means.**

Example 4 **Write a sequence that has two geometric means between 48 and −750.**

This sequence will have the form 48, __?__, __?__, −750.

First, find the common ratio.

$a_n = a_1 r^{n-1}$

$a_4 = a_1 r^3$ *Since there will be four terms in the sequence, $n = 4$.*

$-750 = 48r^3$ *$a_4 = -750$ and $a_1 = 48$*

$\frac{-125}{8} = r^3$ *Divide each side by 48 and simplify.*

$\sqrt[3]{-\frac{125}{8}} = r$ *Take the cube root of each side.*

$-2.5 = r$

Then, determine the geometric sequence.

$a_2 = 48(-2.5)$ or -120 $\qquad$ $a_3 = -120(-2.5)$ or 300

The sequence is 48, −120, 300, −750.

A **geometric series** is the indicated sum of the terms of a geometric sequence. The lists below show some examples of geometric sequences and their corresponding series.

Geometric Sequence	Geometric Series
3, 9, 27, 81, 243	3 + 9 + 27 + 81 + 243
$16, 4, 1, \frac{1}{4}, \frac{1}{16}$	$16 + 4 + 1 + \frac{1}{4} + \frac{1}{16}$
$a_1, a_2, a_3, a_4, \ldots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

To develop a formula for the sum of a finite geometric sequence, S_n, write an expression for S_n and for rS_n, as shown below. Then subtract rS_n from S_n and solve for S_n.

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}$$

$$rS_n = a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n$$

$$S_n - rS_n = a_1 - a_1r^n \quad \textit{Subtract.}$$

$$S_n(1 - r) = a_1 - a_1r^n \quad \textit{Factor.}$$

$$S_n = \frac{a_1 - a_1r^n}{1 - r} \quad \textit{Divide each side by } 1 - r, r \neq 1.$$

Sum of a Finite Geometric Series

The sum of the first n terms of a finite geometric series is given by $S_n = \frac{a_1 - a_1r^n}{1 - r}$.

Example 5 **Find the sum of the first ten terms of the geometric series $16 - 48 + 144 - 432 + \cdots$.**

The formula for the sum of a geometric series can also be written as $S_n = a_1\frac{1 - r^n}{1 - r}$.

First, find the common ratio.

$a_2 \div a_1 = -48 \div 16$ or -3 $\qquad$ $a_4 \div a_3 = -432 \div 144$ or -3

The common ratio is -3.

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

$$S_{10} = \frac{16 - 16(-3)^{10}}{1 - (-3)} \quad n = 10, a_1 = 16, r = -3.$$

$$S_{10} = -236{,}192 \quad \textit{Use a calculator.}$$

The sum of the first ten terms of the series is $-236{,}192$.

Banks and other financial institutions use compound interest to determine earnings in accounts or how much to charge for loans. The formula for compound interest is $A = P\left(1 + \frac{r}{n}\right)^{tn}$, where

A = the account balance,
P = the initial deposit or amount of money borrowed,
r = the annual percentage rate (APR),
n = the number of compounding periods per year, and
t = the time in years.

Suppose at the beginning of each quarter you deposit \$25 in a savings account that pays an APR of 2% compounded quarterly. Most banks post the interest for each quarter on the last day of the quarter. The chart below lists the additions to the account balance as a result of each successive deposit through the rest of the year. Note that $1 + \frac{r}{n} = 1 + \frac{0.02}{4}$ or 1.005.

Date of Deposit	$A = P\left(1 + \frac{r}{n}\right)^{tn}$	1st Year Additions (to the nearest penny)
January 1	$\$25\ (1.005)^4$	\$25.50
April 1	$\$25\ (1.005)^3$	\$25.38
July 1	$\$25\ (1.005)^2$	\$25.25
October 1	$\$25\ (1.005)^1$	\$25.13
Account balance at the end of one year		\$101.26

The chart shows that the first deposit will gain interest through all four compounding periods while the second will earn interest through only three compounding periods. The third and last deposits will earn interest through two and one compounding periods, respectively. The sum of these amounts, \$101.26, is the balance of the account at the end of one year. This sum also represents a finite geometric series where $a_1 = 25.13$, $r = 1.005$, and $n = 4$.

$$25.13 + 25.13(1.005) + 25.13(1.005)^2 + 25.13(1.005)^3$$

Example 6 Real World Application

INVESTMENTS **Hiroshi wants to begin saving money for college. He decides to deposit \$500 at the beginning of each quarter (January 1, April 1, July 1, and October 1) in a savings account that pays an APR of 6% compounded quarterly. The interest for each quarter is posted on the last day of the quarter. Determine Hiroshi's account balance at the end of one year.**

The interest is compounded each quarter. So $n = 4$ and the interest rate per period is 6% ÷ 4 or 1.5%. The common ratio r for the geometric series is then $1 + 0.015$, or 1.015.

The first term a_1 in this series is the account balance at the end of the first quarter. Thus, $a_1 = 500(1.015)$ or 507.5.

Apply the formula for the sum of a geometric series.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_4 = \frac{507.5 - 507.5(1.015)^4}{1 - 1.015} \qquad n = 4,\ r = 1.015$$

$$S_4 \approx 2076.13$$

Hiroshi's account balance at the end of one year is \$2076.13.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** arithmetic and geometric sequences.
2. **Show** that the sequence defined by $a_n = (-3)^{n+1}$ is a geometric sequence.
3. **Explain** why the first term in a geometric sequence must be nonzero.
4. **Find a counterexample** for the statement "The sum of a geometric series cannot be less than its first term."
5. **Determine** whether the given terms form a finite geometric sequence. Write *yes* or *no* and then explain your reasoning.
   a. 3, 6, 18 b. $\sqrt{3}, 3, \sqrt{27}$ c. $x^{-2}, x^{-1}, 1$; $x \neq 1$
6. Refer to Example 3.
   a. **Make a table** to represent the situation. In the first row, put the number of years, and in the second row, put the value of the computers.
   b. **Graph** the numbers in the table. Let years be the *x*-coordinate, let value be the *y*-coordinate, and connect the points.
   c. **Describe** the graph found in part **b**.

Guided Practice

Determine the common ratio and find the next three terms of each geometric sequence.

7. $\frac{2}{3}, 4, 24, \ldots$
8. $2, 3, \frac{9}{2}, \ldots$
9. $1.8, -7.2, 28.8, \ldots$

For Exercises 10–14, assume that each sequence or series is geometric.

10. Find the seventh term of the sequence 7, 2.1, 0.63, ….
11. If $r = 2$ and $a_5 = 24$, find the first term of the sequence.
12. Find the first three terms of the sequence for which $a_4 = 2.5$ and $r = 2$.
13. Write a sequence that has two geometric means between 1 and 27.
14. Find the sum of the first nine terms of the series $0.5 - 1 + 2 - \cdots$.

15. **Investment** Mika Rockwell invests in classic cars. He recently bought a 1978 convertible valued at $20,000. The value of the car is predicted to appreciate at a rate of 3.5% per year. Find the value of the car after 10, 20, and 40 years, assuming that the rate of appreciation remains constant.

EXERCISES

Practice

Determine the common ratio and find the next three terms of each geometric sequence.

16. $10, 2, 0.4, \ldots$
17. $8, -20, 50, \ldots$
18. $\frac{2}{9}, \frac{2}{3}, 2, \ldots$
19. $\frac{3}{4}, \frac{3}{10}, \frac{3}{25}, \ldots$
20. $-7, 3.5, -1.75, \ldots$
21. $3\sqrt{2}, 6, 6\sqrt{2}, \ldots$
22. $9, 3\sqrt{3}, 3, \ldots$
23. $\boldsymbol{i}, -1, -\boldsymbol{i}, \ldots$
24. $t^8, t^5, t^2, \ldots$

www.amc.glencoe.com/self_check_quiz

25. The first term of a geometric sequence is $\frac{a}{b^2}$, and the common ratio is $\frac{b}{a^2}$. Find the next five terms of the geometric sequence.

For Exercises 26–40, assume that each sequence or series is geometric.

26. Find the fifth term of a sequence whose first term is 8 and common ratio is $\frac{3}{2}$.

27. Find the sixth term of the sequence $\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}, \ldots$.

28. Find the seventh term of the sequence 40, 0.4, 0.004, ….

29. Find the ninth term of the sequence $\sqrt{5}, \sqrt{10}, 2\sqrt{5}, \ldots$.

30. If $r = 4$ and $a_6 = 192$, what is the first term of the sequence?

31. If $r = -\sqrt{2}$ and $a_5 = 32\sqrt{2}$, what is the first term of the sequence?

32. Find the first three terms of the sequence for which $a_5 = -6$ and $r = -\frac{1}{3}$.

33. Find the first three terms of the sequence for which $a_5 = 0.32$ and $r = 0.2$.

34. Write a sequence that has three geometric means between 256 and 81.

35. Write a sequence that has two geometric means between −2 and 54.

36. Write a sequence that has one geometric mean between $\frac{4}{7}$ and 7.

37. What is the sum of the first five terms of the series $\frac{5}{3} + 5 + 15 + \cdots$?

38. What is the sum of the first six terms of the series $65 + 13 + 2.6 + \cdots$?

39. Find the sum of the first ten terms of the series $1 - \frac{3}{2} + \frac{9}{4} - \cdots$.

40. Find the sum of the first eight terms of the series $2 + 2\sqrt{3} + 6 + \cdots$.

Applications and Problem Solving

Real World Application

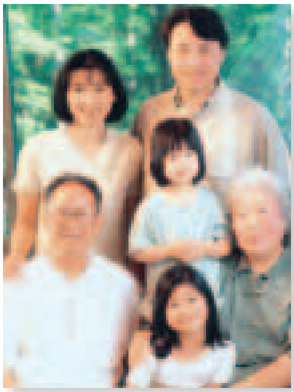
41. **Biology** A cholera bacterium divides every half-hour to produce two complete cholera bacteria.
   a. If an initial colony contains a population of b_0 bacteria, write an equation that will determine the number of bacteria present after t hours.
   b. Suppose a petri dish contains 30 cholera bacteria. Use the equation from part **a** to determine the number of bacteria present 5 hours later.
   c. What assumptions are made in using the formula found in part **a**?

42. **Critical Thinking** Consider the geometric sequence with $a_4 = 4$ and $a_7 = 12$.
   a. Find the common ratio and the first term of the sequence.
   b. Find the 28th term of the sequence.

43. **Consumerism** High Tech Electronics advertises a weekly installment plan for the purchase of a popular brand of big screen TV. The buyer pays $5 at the end of the first week, $5.50 at the end of the second week, $6.05 at the end of the third week, and so on for one year.
   a. What will the payments be at the end of the 10th, 20th, and 40th weeks?
   b. Find the total cost of the TV.
   c. Why is the cost found in part **b** not entirely accurate?

44. **Statistics** A number x is said to be the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$.
   a. Find the harmonic mean of 5 and 8.
   b. 8 is the harmonic mean of 20 and another number. What is the number?

45. **Critical Thinking** In a geometric sequence, $a_1 = -2$ and every subsequent term is defined by $a_n = -3a_{n-1}$, where $n > 1$. Find the nth term in the sequence in terms of n.



46. **Genealogy** Wei-Ling discovers through a research of her Chinese ancestry that one of her fifteenth-generation ancestors was a famous military leader. How many descendants does this ancestor have in the fifteenth-generation, assuming each descendent had an average of 2.5 children?

47. **Personal Finance** Tonisha is about to begin her junior year in high school and is looking ahead to her college career. She estimates that by the time she is ready to enter a university she will need at least \$750 to purchase a computer system that will meet her needs. To avoid purchasing the computer on credit, she opens a savings account earning an APR of 2.4%, compounded monthly, and deposits \$25 at the beginning of each month.
   a. Find the balance of the account at the end of the first month.
   b. If Tonisha continues this deposit schedule for the next two years, will she have enough money in her account to buy the computer system? Explain.
   c. Find the least amount of money Tonisha can deposit each month and still have enough money to purchase the computer.

48. **Critical Thinking** Use algebraic methods to determine which term 6561 is of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$.

Mixed Review

49. **Banking** Gloria Castaneda has \$650 in her checking account. She is closing out the account by writing one check for cash against it each week. The first check is for \$20, the second is for \$25, and so on. Each check exceeds the previous one by \$5. In how many weeks will the balance in Ms. Castaneda account be \$0 if there is no service charge? *(Lesson 12-1)*

50. Find the value of $\log_{11} 265$ using the change of base formula. *(Lesson 11-5)*

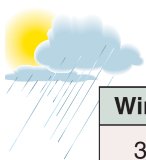
51. Graph the system $xy \geq 2$ and $x - 3y = 2$. *(Lesson 10-8)*

52. Write $3x - 5y + 5 = 0$ in polar form. *(Lesson 9-4)*

53. Write parametric equations of the line $3x + 4y = 5$. *(Lesson 8-6)*

54. If $\csc \theta = 3$ and $0° \leq \theta \leq 90°$, find $\sin \theta$. *(Lesson 7-1)*

55. **Weather** The maximum normal daily temperatures in each season for Lincoln, Nebraska, are given below. Write a sinusoidal function that models the temperatures, using $t = 1$ to represent winter. *(Lesson 6-6)*



Normal Daily Temperatures for Lincoln, Nebraska

Winter	Spring	Summer	Fall
36°	61°	86°	65°

Source: Rand McNally & Company

56. Given $A = 43°$, $b = 20$, and $a = 11$, do these measurements determine one triangle, two triangles, or no triangle? *(Lesson 5-7)*

57. **SAT Practice** **Grid-In** If n and m are integers, and $-(n^2) \leq -\sqrt{49}$ and $m = n + 1$, what is the least possible value of mn?

Extra Practice See p. A48.

12-3

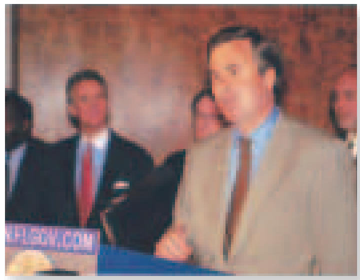
Infinite Sequences and Series

OBJECTIVES

- Find the limit of the terms of an infinite sequence.
- Find the sum of an infinite geometric series.

Real World Application

ECONOMICS On January 28, 1999, Florida governor Jeb Bush proposed a tax cut that would allow the average family to keep an additional $96. The *marginal propensity to consume (MPC)* is defined as the percentage of a dollar by which consumption increases when income rises by a dollar. Suppose the *MPC* for households and businesses in 1999 was 75%. What would be the total amount of money spent in the economy as a result of just one family's tax savings? *This problem will be solved in Example 5.*


Governor Jeb Bush

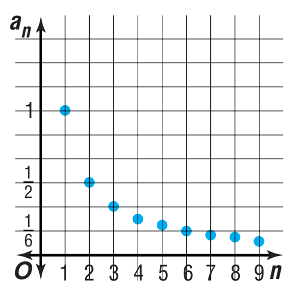
Transaction	Expenditure	Terms of Sequence
1	$96(0.75)^1$	72
2	$96(0.75)^2$	54
3	$96(0.75)^3$	40.50
4	$96(0.75)^4$	30.76
5	$96(0.75)^5$	22.78
⋮	⋮	⋮
10	$96(0.75)^{10}$	5.41
⋮	⋮	⋮
100	$96(0.75)^{100}$	3.08×10^{-11}
⋮	⋮	⋮
500	$96(0.75)^{500}$	3.26×10^{-61}
⋮	⋮	⋮
n	$96(0.75)^n$	ar^n

Study the table at the left. Transaction 1 represents the initial expenditure of $96(0.75) or $72 by a family. The businesses receiving this money, Transaction 2, would in turn spend 75%, and so on. We can write a geometric sequence to model this situation with $a_1 = 72$ and $r = 0.75$. Thus, the geometric sequence representing this situation is

$$72, 54, 40.50, 30.76, 22.78, \ldots.$$

In theory, the sequence above can have infinitely many terms. Thus, it is called an **infinite sequence.** As n increases, the terms of the sequence decrease and get closer and closer to zero. The terms of the modeling sequence will never actually become zero; however, the terms approach zero as n increases without bound.

Consider the infinite sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$, whose nth term, a_n, is $\frac{1}{n}$. Several terms of this sequence are graphed at the right.



Notice that the terms approach a value of 0 as n increases. Zero is called the **limit** of the terms in this sequence.

This limit can be expressed as follows.

$$\lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{∞ is the symbol for infinity.}$$

This is read "the limit of 1 over n, as n approaches infinity, equals zero."

In fact, when any positive power of n appears only in the denominator of a fraction and n approaches infinity, the limit equals zero.

$$\lim_{n \to \infty} \frac{1}{n^r} = 0, \text{ for } r > 0$$

If a general expression for the nth term of a sequence is known, the limit can usually be found by substituting large values for n. Consider the following infinite geometric sequence.

$$7, \frac{7}{4}, \frac{7}{16}, \frac{7}{64}, \frac{7}{256}, \ldots$$

This sequence can be defined by the general expression $a_n = 7\left(\frac{1}{4}\right)^{n-1}$.

$$a_{10} = 7\left(\frac{1}{4}\right)^{10-1} \approx 2.67 \times 10^{-5}$$

$$a_{50} = 7\left(\frac{1}{4}\right)^{50-1} \approx 2.21 \times 10^{-25}$$

$$a_{100} = 7\left(\frac{1}{4}\right)^{100-1} \approx 4.36 \times 10^{-60}$$

Notice that as the value of n increases, the value for a_n appears to approach 0, suggesting $\lim_{n \to \infty} 7\left(\frac{1}{4}\right)^{n-1} = 0$.

Example 1 **Estimate the limit of $\frac{9}{5}, \frac{16}{4}, \frac{65}{27}, \ldots, \frac{7n^2 + 2}{2n^2 + 3n}, \ldots$.**

The 50th term is $\frac{7(50)^2 + 2}{2(50)^2 + 3(50)}$, or about 3.398447.

The 100th term is $\frac{7(100)^2 + 2}{2(100)^2 + 3(100)}$, or about 3.448374.

The 500th term is $7\frac{(500)^2 + 2}{2(500)^2 + 3(500)}$, or about 3.489535.

The 1000th term is $\frac{7(1000)^2 + 2}{2(1000)^2 + 3(1000)}$, or 3.494759.

Notice that as $n \to \infty$, the values appear to approach 3.5, suggesting $\lim_{n \to \infty} \frac{7n^2 + 2}{2n^2 + 3n} = 3.5$.

For sequences with more complicated general forms, applications of the following limit theorems, which we will present without proof, can make the limit easier to find.

Theorems for Limits

If the $\lim_{n\to\infty} a_n$ exists, $\lim_{n\to\infty} b_n$ exists, and c is a constant, then the following theorems are true.

Limit of a Sum	$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
Limit of a Difference	$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
Limit of a Product	$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
Limit of a Quotient	$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, where $\lim_{n\to\infty} b_n \neq 0$
Limit of a Constant	$\lim_{n\to\infty} c_n = c$, where $c_n = c$ for each n

The form of the expression for the nth term of a sequence can often be altered to make the limit easier to find.

Example 2 **Find each limit.**

a. $\lim_{n\to\infty} \frac{(1 + 3n^2)}{n^2}$

$$\lim_{n\to\infty} \frac{(1+3n^2)}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n^2} + 3\right)$$ *Rewrite as the sum of two fractions and simplify.*

$$= \lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} 3$$ *Limit of a Sum*

$$= 0 + 3 \text{ or } 3$$ *$\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\lim_{n\to\infty} 3 = 3$*

Thus, the limit is 3.

Note that the Limit of a Sum theorem only applies here because $\lim_{n\to\infty} \frac{1}{n^2}$ and $\lim_{n\to\infty} 3$ each exist.

b. $\lim_{n\to\infty} \frac{5n^2 + n - 4}{n^2 + 1}$

The highest power of n in the expression is n^2. Divide each term in the numerator and the denominator by n^2. *Why does doing this produce an equivalent expression?*

$$\lim_{n\to\infty} \frac{5n^2 + n - 4}{n^2 + 1} = \lim_{n\to\infty} \frac{\frac{5n^2}{n^2} + \frac{n}{n^2} - \frac{4}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{5 + \frac{1}{n} - \frac{4}{n^2}}{1 + \frac{1}{n^2}}$$ *Simplify.*

$$= \frac{\lim\limits_{n\to\infty} 5 + \lim\limits_{n\to\infty} \frac{1}{n} - \lim\limits_{n\to\infty} 4 \cdot \lim\limits_{n\to\infty} \frac{1}{n^2}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n^2}}$$ *Apply limit theorems.*

$$= \frac{5 + 0 - 4 \cdot 0}{1 + 0} \text{ or } 5$$ *$\lim\limits_{n\to\infty} 5 = 5$, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} 4 = 4$, $\lim\limits_{n\to\infty} 1 = 1$, and $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0a$*

Thus, the limit is 5.

Limits do not exist for all infinite sequences. If the absolute value of the terms of a sequence becomes arbitrarily great or if the terms do not approach a value, the sequence has no limit. Example 3 illustrates both of these cases.

Example 3 **Find each limit.**

a. $\lim\limits_{n\to\infty} \frac{2 + 5n + 4n^2}{2n}$

$$\lim_{n\to\infty} \frac{2 + 5n + 4n^2}{2n} = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{5}{2} + 2n\right)$$ *Simplify.*

Note that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\lim\limits_{n\to\infty} \frac{5}{2} = \frac{5}{2}$, but $2n$ becomes increasingly large as n approaches infinity. Therefore, the sequence has no limit.

b. $\lim\limits_{n\to\infty} \frac{(-1)^n n}{8n + 1}$

Begin by rewriting $\frac{(-1)^n n}{8n + 1}$ as $(-1)^n \cdot \frac{n}{8n + 1}$. Now find $\lim\limits_{n\to\infty} \frac{n}{8n + 1}$.

$$\lim_{n\to\infty} \frac{n}{8n + 1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{8n}{n} + \frac{1}{n}}$$ *Divide the numerator and denominator by n.*

$$= \lim_{n\to\infty} \frac{1}{8 + \frac{1}{n}}$$ *Simplify.*

$$= \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 8 + \lim\limits_{n\to\infty} \frac{1}{n}}$$ *Apply limit theorems.*

$$= \frac{1}{8}$$ *$\lim\limits_{n\to\infty} 1 = 1$, $\lim\limits_{n\to\infty} 8 = 8$, and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$*

When n is even, $(-1)^n = 1$. When n is odd, $(-1)^n = -1$. Thus, the odd-numbered terms of the sequence described by $\frac{(-1)^n n}{8n + 1}$ approach $-\frac{1}{8}$, and the even-numbered terms approach $+\frac{1}{8}$. Therefore, the sequence has no limit.

An **infinite series** is the indicated sum of the terms of an infinite sequence. Consider the series $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$. Since this is a geometric series, you can find the sum of the first 100 terms by using the formula $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r = \frac{1}{5}$.

$$S_{100} = \frac{\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}}{1 - \frac{1}{5}}$$

$$= \frac{\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}}{\frac{4}{5}}$$

$$= \frac{5}{4}\left[\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}\right] \text{ or } \frac{1}{4} - \frac{1}{4}\left(\frac{1}{5}\right)^{100}$$

Since $\left(\frac{1}{5}\right)^{100}$ is very close to 0, S_{100} is nearly equal to $\frac{1}{4}$. No matter how many terms are added, the sum of the infinite series will never exceed $\frac{1}{4}$, and the difference from $\frac{1}{4}$ gets smaller as $n \to \infty$. Thus, $\frac{1}{4}$ is the sum of the infinite series.

Sum of an Infinite Series

If S_n is the sum of the first n terms of a series, and S is a number such that $S - S_n$ approaches zero as n increases without bound, then the sum of the infinite series is S.

$$\lim_{n \to \infty} S_n = S$$

If the sequence of partial sums S_n has a limit, then the corresponding infinite series has a sum, and the nth term a_n of the series approaches 0 as $n \to \infty$. If $\lim_{n \to \infty} a_n \neq 0$, the series has no sum. If $\lim_{n \to \infty} a_n = 0$, the series may or may not have a sum.

The formula for the sum of the first n terms of a geometric series can be written as follows.

$$S_n = a_1 \frac{(1 - r^n)}{1 - r}, r \neq 1$$

Recall that $|r| < 1$ means $-1 < r < 1$.

Suppose $n \to \infty$; that is, the number of terms increases without limit. If $|r| > 1$, r^n increases without limit as $n \to \infty$. However, when $|r| < 1$, r^n approaches 0 as $n \to \infty$. Under this condition, the above formula for S_n approaches a value of $\frac{a_1}{1 - r}$.

Sum of an Infinite Geometric Series

The sum S of an infinite geometric series for which $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

Example 4 **Find the sum of the series $21 - 3 + \frac{3}{7} - \cdots$.**

In the series, $a_1 = 21$ and $r = -\frac{1}{7}$. Since $|r| < 1$, $S = \frac{a_1}{1 - r}$.

$$S = \frac{a_1}{1 - r}$$

$$= \frac{21}{1 - \left(-\frac{1}{7}\right)} \quad a_1 = 21,\ r = -\frac{1}{7}$$

$$= \frac{147}{8} \text{ or } 18\frac{3}{8}$$

The sum of the series is $18\frac{3}{8}$.

In economics, finding the sum of an infinite series is useful in determining the overall effect of economic trends.

Example 5 Real World Application

ECONOMICS **Refer to the application at the beginning of the lesson. What would be the total amount of money spent in the economy as a result of just one family's tax savings?**

For the geometric series modeling this situation, $a_1 = 72$ and $r = 0.75$.

Since $|r| < 1$, the sum of the series is equal to $\frac{a_1}{1 - r}$.

$$S = \frac{a_1}{1 - r}$$

$$= \frac{72}{1 - 0.75} \text{ or } 288$$

Therefore, the total amount of money spent is $288.

You can use what you know about infinite series to write repeating decimals as fractions. The first step is to write the repeating decimal as an infinite geometric series.

Example 6 **Write $0.\overline{762}$ as a fraction.**

$$0.\overline{762} = \frac{762}{1000} + \frac{762}{1{,}000{,}000} + \frac{762}{1{,}000{,}000{,}000} + \cdots$$

In this series, $a_1 = \frac{762}{1000}$ and $r = \frac{1}{1000}$.

(continued on the next page)

$$S = \frac{a_1}{1 - r}$$

$$= \frac{\frac{762}{1000}}{1 - \frac{1}{1000}}$$

$$= \frac{762}{999} \text{ or } \frac{254}{333}$$

Thus, $0.762762 \cdots = \frac{254}{333}$. *Check this with a calculator.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. Consider the sequence given by the general expression $a_n = \frac{n - 1}{n}$.
   a. **Graph** the first ten terms of the sequence with the term number on the x-axis and the value of the term on the y-axis.
   b. **Describe** what happens to the value of a_n as n increases.
   c. **Make a conjecture** based on your observation in part **a** as to the limit of the sequence as n approaches infinity.
   d. **Apply** the techniques presented in the lesson to evaluate $\lim_{n \to \infty} \frac{n - 1}{n}$. How does your answer compare to your conjecture made in part **c**?
2. Consider the infinite geometric sequence given by the general expression r^n.
   a. **Determine** the limit of the sequence for $r = \frac{1}{2}$, $r = \frac{1}{4}$, $r = 1$, $r = 2$, and $r = 5$.
   b. **Write** a general rule for the limit of the sequence, placing restrictions on the value of r.
3. **Give an example** of an infinite geometric series having no sum.
4. **You Decide** Tyree and Zonta disagree on whether the infinite sequence described by the general expression $2n - 3$ has a limit. Tyree says that after dividing by the highest-powered term, the expression simplifies to $2 - \frac{3}{n}$, which has a limit of 2 as n approaches infinity. Zonta says that the sequence has no limit. Who is correct? Explain.

Guided Practice

Find each limit, or state that the limit does not exist and explain your reasoning.

5. $\lim_{n \to \infty} \frac{1}{5^n}$
6. $\lim_{n \to \infty} \frac{5 - n^2}{2n}$
7. $\lim_{n \to \infty} \frac{3n - 6}{7n}$

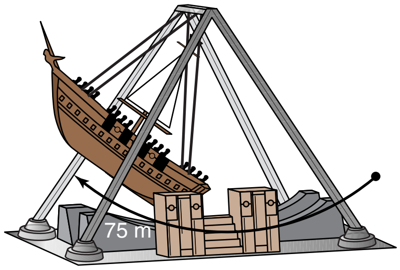
Write each repeating decimal as a fraction.

8. $0.\overline{7}$
9. $5.\overline{126}$

Find the sum of each infinite series, or state that the sum does not exist and explain your reasoning.

10. $-6 + 3 - \frac{3}{2} + \cdots$ **11.** $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \cdots$ **12.** $\sqrt{3} + 3 + \sqrt{27} + \cdots$

13. Entertainment Pete's Pirate Ride operates like the bob of a pendulum. On its longest swing, the ship travels through an arc 75 meters long. Each successive swing is two-fifths the length of the preceding swing. If the ride is allowed to continue without intervention, what is the total distance the ship will travel before coming to rest?



EXERCISES

Practice

Find each limit, or state that the limit does not exist and explain your reasoning.

14. $\lim_{n\to\infty} \frac{7-2n}{5n}$ **15.** $\lim_{n\to\infty} \frac{n^3-2}{n^2}$ **16.** $\lim_{n\to\infty} \frac{6n^2+5}{3n^2}$

17. $\lim_{n\to\infty} \frac{9n^3+5n-2}{2n^3}$ **18.** $\lim_{n\to\infty} \frac{(3n+4)(1-n)}{n^2}$ **19.** $\lim_{n\to\infty} \frac{8n^2+5n+2}{3+2n}$

20. $\lim_{n\to\infty} \frac{4-3n+n^2}{2n^3-3n^2+5}$ **21.** $\lim_{n\to\infty} \frac{n}{3^n}$ **22.** $\lim_{n\to\infty} \frac{(-2)^n n}{4+n}$

23. Find the limit of the sequence described by the general expression $\frac{5n+(-1)^n}{n^2}$, or state that the limit does not exist. Explain your reasoning.

Write each repeating decimal as a fraction.

24. $0.\overline{4}$ **25.** $0.\overline{51}$ **26.** $0.\overline{370}$

27. $6.\overline{259}$ **28.** $0.\overline{15}$ **29.** $0.2\overline{63}$

30. Explain why the sum of the series $0.2 + 0.02 + 0.002 + \cdots$ exists. Then find the sum.

Find the sum of each series, or state that the sum does not exist and explain your reasoning.

31. $16 + 12 + 9 + \cdots$ **32.** $5 + 7.5 + 11.25 + \cdots$ **33.** $10 + 5 + 2.5 + \cdots$

34. $6 + 5 + 4 + \cdots$ **35.** $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \cdots$ **36.** $-\frac{2}{3} + \frac{1}{9} - \frac{1}{54} + \cdots$

37. $\frac{6}{5} + \frac{4}{5} + \frac{8}{15} + \cdots$ **38.** $\sqrt{5} + 1 + \frac{\sqrt{5}}{5} + \cdots$ **39.** $8 - 4\sqrt{3} + 6 - \cdots$

Applications and Problem Solving

40. Physics A basketball is dropped from a height of 35 meters and bounces $\frac{2}{5}$ of the distance after each fall.

a. Find the first five terms of the infinite series representing the vertical distance traveled by the ball.

b. What is the total vertical distance the ball travels before coming to rest? (*Hint:* Rewrite the series found in part **a** as the sum of two infinite geometric series.)

41. Critical Thinking Consider the sequence whose nth term is described by $\frac{n^2}{2n+1} - \frac{n^2}{2n-1}$.

a. Explain why $\lim_{n\to\infty}\left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right) \neq \lim_{n\to\infty}\frac{n^2}{2n+1} - \lim_{n\to\infty}\frac{n^2}{2n-1}$.

b. Find $\lim_{n\to\infty}\left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right)$.

42. Engineering Francisco designs a toy with a rotary flywheel that rotates at a maximum speed of 170 revolutions per minute. Suppose the flywheel is operating at its maximum speed for one minute and then the power supply to the toy is turned off. Each subsequent minute thereafter, the flywheel rotates two-fifths as many times as in the preceding minute. How many complete revolutions will the flywheel make before coming to a stop?

43. Critical Thinking Does $\lim_{n\to\infty} \cos \frac{n\pi}{2}$ exist? Explain.

44. Medicine A certain drug developed to fight cancer has a half-life of about 2 hours in the bloodstream. The drug is formulated to be administered in doses of D milligrams every 6 hours. The amount of each dose has yet to be determined.

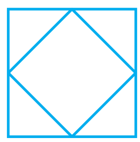
a. What fraction of the first dose will be left in the bloodstream before the second dose is administered?

b. Write a general expression for the geometric series that models the number of milligrams of drug left in the bloodstream after the nth dose.

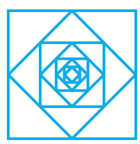
c. About what amount of medicine is present in the bloodstream for large values of n?

d. A level of more than 350 milligrams of this drug in the bloodstream is considered toxic. Find the largest possible dose that can be given repeatedly over a long period of time without harming the patient.

45. Geometry If the midpoints of a square are joined by straight lines, the new figure will also be a square.

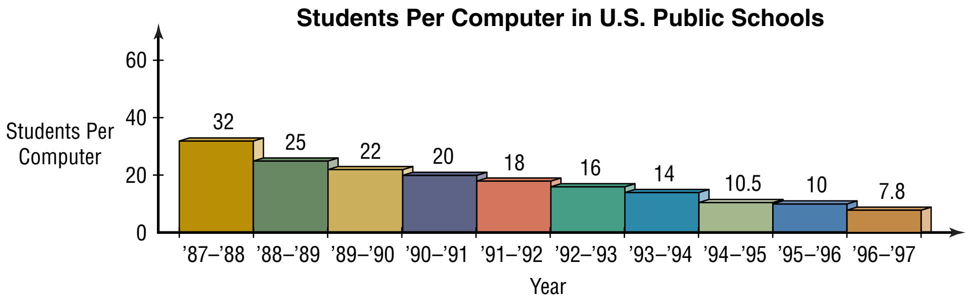


a. If the original square has a perimeter of 20 feet, find the perimeter of the new square. (*Hint:* Use the Pythagorean Theorem.)



b. If this process is continued to form a sequence of "nested" squares, what will be the sum of the perimeters of all the squares?

46. Technology Since the mid-1980s, the number of computers in schools has steadily increased. The graph below shows the corresponding decline in the student-computer ratio.



Source: *QED's Technology in Public Schools*, 16th Edition

Another publication states that the average number of students per computer in U.S. public schools can be estimated by the sequence model $a_n = 35.812791(0.864605)^n$, *for* $n = 1, 2, 3, \ldots$, with the 1987-1988 school year corresponding to $n = 1$.

a. Find the first ten terms of the model. Round your answers to the nearest tenth.

b. Use the model to estimate the average number of students having to share a computer during the 1995-1996 school year. How does this estimate compare to the actual data given in the graph?

c. Make a prediction as to the average number of students per computer for the 2000-2001 school year.

d. Does this sequence approach a limit? If so, what is the limit?

e. Realistically, will the student computer ratio ever reach this limit? Explain.

Mixed Review

47. The first term of a geometric sequence is -3, and the common ratio is $\frac{2}{3}$. Find the next four terms of the sequence. *(Lesson 12-2)*

48. Find the 16th term of the arithmetic sequence for which $a_1 = 1.5$ and $d = 0.5$. *(Lesson 12-1)*

49. Name the coordinates of the center, foci, and vertices, and the equation of the asymptotes of the hyperbola that has the equation $x^2 - 4y^2 - 12x - 16y = -16$. *(Lesson 10-4)*

50. Graph $r = 6 \cos 3\theta$. *(Lesson 9-2)*

51. Navigation A ship leaving port sails for 125 miles in a direction 20° north of due east. Find the magnitude of the vertical and horizontal components. *(Lesson 8-1)*

52. Use a half-angle identity to find the exact value of cos 112.5°. *(Lesson 7-4)*

53. Graph $y = \cos x$ on the interval $-180° \leq x \leq 360°$. *(Lesson 6-3)*

54. List all possible rational zeros of the function $f(x) = 8x^3 + 3x - 2$. *(Lesson 4-4)*

55. SAT/ACT Practice If $a = 4b + 26$, and b is a positive integer, then a could be divisible by all of the following EXCEPT

A 2 **B** 4 **C** 5 **D** 6 **E** 7

Extra Practice See p. A49.

GRAPHING CALCULATOR EXPLORATION

12-3B Continued Fractions

An Extension of Lesson 12-3

OBJECTIVE

- Explore sequences generated by continued fractions.

An expression of the following form is called a *continued fraction*.

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}$$

By using only a finite number of "decks" and values of a_n and b_n that follow regular patterns, you can often obtain a sequence of terms that approaches a limit, which can be represented by a simple expression. For example, if all of the numbers a_n and b_n are equal to 1, then the continued fraction gives rise to the following sequence.

$$1,\ 1 + \frac{1}{1},\ 1 + \cfrac{1}{1 + \cfrac{1}{1}},\ 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}},\ \ldots$$

It can be shown that the terms of this sequence approach the limit $\frac{1 + \sqrt{5}}{2}$. This number is often called the *golden ratio*.

The golden ratio is closely related to the Fibonacci sequence, which you will learn about in Lesson 12-7.

Now consider the following more general sequence.

$$A,\ A + \frac{1}{A},\ A + \cfrac{1}{A + \cfrac{1}{A}},\ A + \cfrac{1}{A + \cfrac{1}{A + \cfrac{1}{A}}},\ \ldots$$

To help you visualize what this sequence represents, suppose $A = 5$. The sequence becomes $5,\ 5 + \frac{1}{5},\ 5 + \cfrac{1}{5 + \cfrac{1}{5}},\ 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5}}},\ \ldots$ or $5, \frac{26}{5}, \frac{135}{26}, \frac{701}{135}, \ldots$.

A calculator approximation of this sequence is 5, 5.2, 5.192307692, 5,192592593, … .

interNET CONNECTION

Graphing Calculator Programs
To download this graphing calculator program, visit our website at **www.amc.glencoe.com**

Each term of the sequence is the sum of A and the reciprocal of the previous term. The program at the right calculates the value of the nth term of the above sequence for $n \geq 3$ and a specified value of A.

When you run the program it will ask you to input values for A and N.

```
PROGRAM: CFRAC
: Prompt A
: Disp "INPUT TERM"
: Disp "NUMBER N, N ≥ 3"
: Prompt N
: 1 → K
: A + 1/A → C
: Lbl 1
: A + 1/C → C
: K + 1 → K
: If K < N – 1
: Then: Goto 1
: Else: Disp C
```

TRY THESE **Enter the program into your calculator and use it for the exercises that follow.**

1. What output is given when the program is executed for $A = 1$ and $N = 10$?

2. With $A = 1$, determine the least value of N necessary to obtain an output that agrees with the calculator's nine decimal approximation of $\frac{1 + \sqrt{5}}{2}$.

3. Use algebra to show that the continued fraction $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$ has a value of $\frac{1 + \sqrt{5}}{2}$. (*Hint:* If $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$, then $x = 1 + \frac{1}{x}$. Solve this last equation for x.)

4. Find the exact value of $3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}$.

5. Execute the program with $A = 3$ and $N = 40$. How does this output compare to the decimal approximation of the expression found in Exercise 4?

6. Find a radical expression for $A + \cfrac{1}{A + \cfrac{1}{A + \cdots}}$.

7. Write a modified version of the program that calculates the nth term of the following sequence for $n \geq 3$.

$$A,\ A + \frac{B}{2A},\ A + \cfrac{B}{2A + \cfrac{B}{2A}},\ A + \cfrac{B}{2A + \cfrac{B}{2A + \cfrac{B}{2A}}},\ \ldots$$

8. Choose several positive integer values for A and B and compare the program output with the decimal approximation of $\sqrt{A^2 + B}$ for several values of n, for $n \geq 3$. Describe your observations.

9. Use algebra to show that for $A > 0$ and $B > 0$, $A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}$ has a value of $\sqrt{A^2 + B}$.

$$\left(\textit{Hint: } \text{If } x = A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}, \text{ then } x + A = 2A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}.\right)$$

WHAT DO YOU THINK?

10. If you execute the original program for $A = 1$ and $N = 20$ and then execute it for $A = -1$ and $N = 20$, how will the two outputs compare?

11. What values can you use for A and B in the program for Exercise 7 in order to approximate $\sqrt{15}$?

12-4

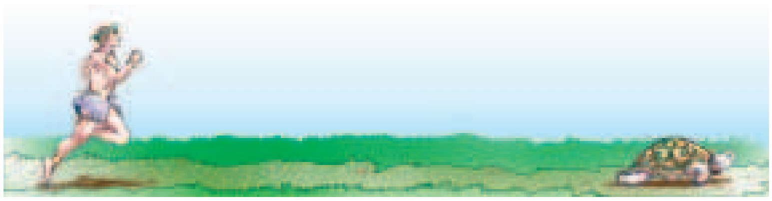
Convergent and Divergent Series

OBJECTIVE

- Determine whether a series is convergent or divergent.

Real World Application

HISTORY The Greek philosopher Zeno of Elea (c. 490–430 B.C.) proposed several perplexing riddles, or paradoxes. One of Zeno's paradoxes involves a race on a 100-meter track between the mythological Achilles and a tortoise. Zeno claims that even though Achilles can run twice as fast as the tortoise, if the tortoise is given a 10-meter head start, Achilles will never catch him. Suppose Achilles runs 10 meters per second and the tortoise a remarkable 5 meters per second. By the time Achilles has reached the 10-meter mark, the tortoise will be at 15 meters. By the time Achilles reaches the 15-meter mark, the tortoise will be at 17.5 meters, and so on. Thus, Achilles is always behind the tortoise and never catches up.



Is Zeno correct? Let us look at the distance between Achilles and the tortoise after specified amounts of time have passed. Notice that the distance between the two contestants will be zero as n approaches infinity since $\lim_{n \to \infty} \frac{10}{2^n} = 0$.

Time (seconds)	Distance Apart (meters)
0	10
1	$\frac{10}{2} = 5$
$1 + \frac{1}{2} = \frac{3}{2}$	$\frac{10}{4} = 2.5$
$1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$	$\frac{10}{8} = 1.25$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$ $\vdots$	$\frac{10}{16} = 0.625$ $\vdots$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$	$\frac{10}{2^n}$

To disprove Zeno's conclusion that Achilles will never catch up to the tortoise, we must show that there is a time value for which this 0 difference can be achieved. In other words, we need to show that the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has a sum, or limit. *This problem will be solved in Example 5.*

Starting with a time of 1 second, the partial sums of the time series form the sequence $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{18}, \ldots$. As the number of terms used for the partial sums increases, the value of the partial sums also increases. If this sequence of partial sums approaches a limit, the related infinite series is said to **converge.** If this sequence of partial sums does not have a limit, then the related infinite series is said to **diverge.**

Convergent and Divergent Series	If an infinite series has a sum, or limit, the series is convergent. If a series is not convergent, it is divergent.

Example 1 **Determine whether each arithmetic or geometric series is convergent or divergent.**

There are many series that begin with the first few terms shown in this example. In this chapter, always assume that the expression for the general term is the simplest one possible.

a. $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

This is a geometric series with $r = -\frac{1}{2}$. Since $|r| < 1$, the series has a limit. Therefore, the series is convergent.

b. $2 + 4 + 8 + 16 + \cdots$

This is a geometric series with $r = 2$. Since $|r| > 1$, the series has no limit. Therefore, the series is divergent.

c. $10 + 8.5 + 7 + 5.5 + \cdots$

This is an arithmetic series with $d = -1.5$. Arithmetic series do not have limits. Therefore, the series is divergent.

When a series is neither arithmetic nor geometric, it is more difficult to determine whether the series is convergent or divergent. Several different techniques can be used. One test for convergence is the **ratio test.** This test can only be used when all terms of a series are positive. The test depends upon the ratio of consecutive terms of a series, which must be expressed in general form.

Ratio Test	Let a_n and a_{n+1} represent two consecutive terms of a series of positive terms. Suppose $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and that $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$. The series is convergent if $r < 1$ and divergent if $r > 1$. If $r = 1$, the test provides no information.

The ratio test is especially useful when the general form for the terms of a series contains powers.

Example 2 **Use the ratio test to determine whether each series is convergent or divergent.**

a. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$

First, find a_n and a_{n+1}. $\quad a_n = \frac{n}{2^n} \quad$ and $\quad a_{n+1} = \frac{n+1}{2^{n+1}}$

Then use the ratio test. $\quad r = \lim\limits_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}$

(continued on the next page)

$$r = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$$ *Multiply by the reciprocal of the divisor.*

$$r = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n}$$ $\frac{2^n}{2^{n+1}} = \frac{1}{2}$

$$r = \lim_{n\to\infty} \frac{1}{2} \cdot \lim_{n\to\infty} \frac{n+1}{n}$$ *Limit of a Product*

$$r = \frac{1}{2} \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{1}$$ *Divide by the highest power of n and then apply limit theorems.*

$$r = \frac{1}{2} \cdot \frac{1+0}{1} \text{ or } \frac{1}{2}$$ Since $r < 1$, the series is convergent.

b. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

$$a_n = \frac{n}{n+1} \quad \text{and} \quad a_{n+1} = \frac{n+1}{(n+1)+1} \text{ or } \frac{n+1}{n+2}$$

$$r = \lim_{n\to\infty} \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}}$$

$$r = \lim_{n\to\infty} \frac{n^2 + 2n + 1}{n^2 + 2n}$$ $\frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$

$$r = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}}$$ *Divide by the highest power of n and apply limit theorems.*

$$r = \frac{1+0+0}{1+0} \text{ or } 1$$ Since $r = 1$, the test provides no information.

The ratio test is also useful when the general form of the terms of a series contains products of consecutive integers.

Example 3 **Use the ratio test to determine whether the series**

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$ **is convergent or divergent.**

First find the nth term and $(n+1)$th term. Then, use the ratio test.

$$a_n = \frac{1}{1 \cdot 2 \cdot \cdots \cdot n} \quad \text{and} \quad a_{n+1} = \frac{1}{1 \cdot 2 \cdot \cdots \cdot (n+1)}$$

$$r = \lim_{n\to\infty} \frac{\frac{1}{1 \cdot 2 \cdot \cdots \cdot (n+1)}}{\frac{1}{1 \cdot 2 \cdot \cdots \cdot n}}$$

$$r = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot (n+1)}$$ *Note that $1 \cdot 2 \cdot \cdots \cdot (n+1) = 1 \cdot 2 \cdot \cdots \cdot n \cdot (n+1)$.*

$$r = \lim_{n\to\infty} \frac{1}{n+1} \text{ or } 0$$ *Simplify and apply limit theorems.*

Since $r < 1$, the series is convergent.

When the ratio test does not determine if a series is convergent or divergent, other methods must be used.

Example 4 **Determine whether the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is convergent or divergent.**

Suppose the terms are grouped as follows. Beginning after the second term, the number of terms in each successive group is doubled.

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

Notice that the first enclosed expression is greater than $\frac{1}{2}$, and the second is equal to $\frac{1}{2}$. Beginning with the third expression, each sum of enclosed terms is greater than $\frac{1}{2}$. Since there are an unlimited number of such expressions, the sum of the series is unlimited. Thus, the series is divergent.

A series can be compared to other series that are known to be convergent or divergent. The following list of series can be used for reference.

Summary of Series for Reference

1. **Convergent:** $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} + \cdots, |r| < 1$
2. **Divergent:** $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} + \cdots, |r| > 1$
3. **Divergent:** $a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots$
4. **Divergent:** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$ *This series is known as the harmonic series.*
5. **Convergent:** $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, p > 1$

If a series has all positive terms, the **comparison test** can be used to determine whether the series is convergent or divergent.

Comparison Test

- A series of positive terms is convergent if, for $n > 1$, each term of the series is equal to or less than the value of the corresponding term of some convergent series of positive terms.
- A series of positive terms is divergent if, for $n > 1$, each term of the series is equal to or greater than the value of the corresponding term of some divergent series of positive terms.

Example 5 **Use the comparison test to determine whether the following series are convergent or divergent.**

a. $\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \cdots$

The general term of this series is $\frac{4}{2n+3}$. The general term of the divergent series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is $\frac{1}{n}$. Since for all $n > 1$, $\frac{4}{2n+3} > \frac{1}{n}$, the series $\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \cdots$ is also divergent.

b. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

The general term of the series is $\frac{1}{(2n - 1)^2}$. The general term of the convergent series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is $\frac{1}{n^2}$. Since $\frac{1}{(2n - 1)^2} \leq \frac{1}{n^2}$ for all n, the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$ is also convergent.

With a better understanding of convergent and divergent infinite series, we are now ready to tackle Zeno's paradox.

Example 6 Real World Application

HISTORY **Refer to the application at the beginning of the lesson. To disprove Zeno's conclusion that Achilles will never catch up to the tortoise, we must show that the infinite time series $1 + 0.5 + 0.25 + \cdots$ has a limit.**

To show that the series $1 + 0.5 + 0.25 + \cdots$ has a limit, we need to show that the series is convergent.

The general term of this series is $\frac{1}{2^n}$. Try using the ratio test for convergence of a series.

$$a_n = \frac{1}{2^n} \qquad \text{and} \qquad a_{n+1} = \frac{1}{2^{n+1}}$$

$$r = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}$$

$$= \frac{1}{2} \qquad \frac{1}{2^{n+1}} \cdot \frac{2^n}{1} = \frac{1}{2}$$

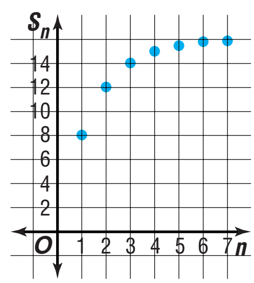
Since $r < 1$, the series converges and therefore has a sum. Thus, there is a time value for which the distance between Achilles and the tortoise will be zero. *You will determine how long it takes him to do so in Exercise 34.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. a. **Write** an example, of an infinite geometric series in which $|r| > 1$.
   b. **Determine** the 25th, 50th, and 100th terms of your series.
   c. **Identify** the sum of the first 25, 50, and 100 terms of your series.
   d. **Explain** why this type of infinite geometric series does not converge.
2. **Estimate** the sum S_n of the series whose partial sums are graphed at the right.



3. Consider the infinite series $\frac{1}{3} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \frac{4^2}{3^4} + \cdots$.

a. **Sketch** a graph of the first eight partial sums of this series.

b. **Make** a conjecture based on the graph found in part **a** as to whether the series is convergent or divergent.

c. **Determine** a general term for this series.

d. **Write a convincing argument** using the general term found in part **c** to support the conjecture you made in part **b**.

4. *Math Journal* **Make a list** of the methods presented in this lesson and in the previous lesson for determining convergence or divergence of an infinite series. Be sure to indicate any restrictions on a method's use. Then number your list as to the order in which these methods should be applied.

Guided Practice

Use the ratio test to determine whether each series is *convergent* or *divergent.*

5. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$

6. $\frac{3}{4} + \frac{7}{8} + \frac{11}{12} + \frac{15}{16} + \cdots$

7. Use the comparison test to determine whether the series $\frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots$ is *convergent* or *divergent.*

Determine whether each series is *convergent* or *divergent.*

8. $\frac{1}{4} + \frac{5}{16} + \frac{3}{8} + \frac{7}{16} + \cdots$

9. $\frac{1}{2+1^2} + \frac{1}{2+2^2} + \frac{1}{2+3^2} + \cdots$

10. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$

11. $4 + 3 + \frac{9}{4} + \cdots$

12. **Ecology** An underground storage container is leaking a toxic chemical. One year after the leak began, the chemical had spread 1500 meters from its source. After two years, the chemical had spread 900 meters more, and by the end of the third year, it had reached an additional 540 meters.

a. If this pattern continues, how far will the spill have spread from its source after 10 years?

b. Will the spill ever reach the grounds of a school located 4000 meters away from the source? Explain.

EXERCISES

Practice

Use the ratio test to determine whether each series is *convergent* or *divergent.*

13. $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots$

14. $\frac{2}{5} + \frac{4}{10} + \frac{8}{15} + \cdots$

15. $2 + \frac{4}{2^2} + \frac{8}{3^2} + \frac{16}{4^2} + \cdots$

16. $\frac{2}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{2}{4 \cdot 5} + \cdots$

17. $1 + \frac{3}{1 \cdot 2 \cdot 3} + \frac{5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$

18. $5 + \frac{5^2}{1 \cdot 2} + \frac{5^3}{1 \cdot 2 \cdot 3} + \cdots$

19. Use the ratio test to determine whether the series $\frac{2 \cdot 4}{2} + \frac{4 \cdot 6}{4} + \frac{6 \cdot 8}{8} + \frac{8 \cdot 10}{16} + \cdots$ is convergent or divergent.

Use the comparison test to determine whether each series is *convergent* or *divergent*.

20. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$

21. $\frac{1}{2} + \frac{1}{9} + \frac{1}{28} + \frac{1}{65} + \cdots$

22. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

23. $\frac{5}{3} + \frac{5}{4} + 1 + \frac{5}{6} + \cdots$

24. Use the comparison test to determine whether the series $\frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} \cdots$ is *convergent* or *divergent*.

Determine whether each series is *convergent* or *divergent*.

25. $\frac{1}{2} - \frac{3}{8} + \frac{9}{32} - \cdots$

26. $3 + \frac{5}{3} + \frac{7}{5} + \cdots$

27. $\frac{1}{5 + 1^2} + \frac{1}{5 + 2^2} + \frac{1}{5 + 3^2} + \cdots$

28. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$

29. $\frac{4\pi}{3} + \frac{5\pi}{6} + \frac{\pi}{3} + \cdots$

30. $\frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \frac{7}{32} + \cdots$

Applications and Problem Solving

31. Economics The MagicSoft software company has a proposal to the city council of Alva, Florida, to relocate there. The proposal claims that the company will generate \$3.3 million for the local economy by the \$1 million in salaries that will be paid. The city council estimates that 70% of the salaries will be spent in the local community, and 70% of that money will again be spent in the community, and so on.

a. According to the city council's estimates, is the claim made by MagicSoft accurate? Explain.

b. What is the correct estimate of the amount generated to the local economy?

32. Critical Thinking Give an example of a series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ that diverges, but when its terms are squared, the resulting series $a_1{}^2 + a_2{}^2 + a_3{}^2 + \cdots + a_n{}^2 + \cdots$ converges.

33. Cellular Growth Leticia Cox is a biochemist. She is testing two different types of drugs that induce cell growth. She has selected two cultures of 1000 cells each. To culture A, she administers a drug that raises the number of cells by 200 each day and every day thereafter. Culture B gets a drug that increases cell growth by 8% each day and everyday thereafter.

a. Assuming no cells die, how many cells will have grown in each culture by the end of the seventh day?

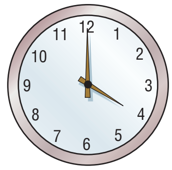
b. At the end of one month's time, which drug will prove to be more effective in promoting cell growth? Explain.

34. Critical Thinking Refer to Example 6 of this lesson. The sequence of partial sums, $S_1, S_2, S_3, \ldots, S_n, \ldots$, for the time series is $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$.

a. Find a general expression for the nth term of this sequence.

b. To determine how long it takes for Achilles to catch-up to the tortoise, find the sum of the infinite time series. (*Hint:* Recall from the definition of the sum S of an infinite series that $\lim_{n \to \infty} S_n = S$.)

35. Clocks The hour and minute hands of a clock travel around its face at different speeds, but at certain times of the day, the two hands coincide. In addition to noon and midnight, the hands also coincide at times occurring between the hours. According the figure at the right, it is 4:00.



a. When the minute hand points to 4, what fraction of the distance between 4 and 5 will the hour hand have traveled?

b. When the minute hand reaches the hour hand's new position, what additional fraction will the hour hand have traveled?

c. List the next two terms of this series representing the distance traveled by the hour hand as the minute hand "chases" its position.

d. At what time between the hours of 4 and 5 o'clock will the two hands coincide?

Mixed Review

36. Evaluate $\lim_{n \to \infty} \frac{4n^2 + 5}{3n^2 - 2n}$. *(Lesson 12-3)*

37. Find the ninth term of the geometric sequence $\sqrt{2}, 2, 2\sqrt{2}, \ldots$. *(Lesson 12-2)*

38. Form an arithmetic sequence that has five arithmetic means between -11 and 19. *(Lesson 12-1)*

39. Solve $45.9 = e^{0.075t}$ *(Lesson 11-6)*

40. Navigation A submarine sonar is tracking a ship. The path of the ship is recorded as $6 = 12r \cos(\theta - 30°)$. Find the linear equation of the path of the ship. *(Lesson 9-4)*

41. Find an ordered pair that represents $\overrightarrow{AB}$ for $A(8, -3)$ and $B(5, -1)$. *(Lesson 8-2)*

42. SAT/ACT Practice How many numbers from 1 to 200 inclusive are equal to the cube of an integer?

A one **B** two **C** three **D** four **E** five

MID-CHAPTER QUIZ

1. Find the 19th term in the sequence for which $a_1 = 11$ and $d = -2$. (Lesson 12-1)

2. Find S_{20} for the arithmetic series for which $a_1 = -14$ and $d = 6$. (Lesson 12-1)

3. Form a sequence that has two geometric means between 56 and 189. (Lesson 12-2)

4. Find the sum of the first eight terms of the series $3 - 6 + 12 - \cdots$. (Lesson 12-2)

5. Find $\lim_{n \to \infty} \frac{n^2 + 2n - 5}{n^2 - 1}$ or explain why the limit does not exist. (Lesson 12-3)

6. Recreation A bungee jumper rebounds 55% of the height jumped. If a bungee jump is made using a cord that stretches 250 feet, find the total distance traveled by the jumper before coming to rest. (Lesson 12-3)

7. Find the sum of the following series.
$\frac{1}{25} + \frac{1}{250} + \frac{1}{2500} + \cdots$. (Lesson 12-3)

Determine whether each series is *convergent* or *divergent*. (Lesson 12-4)

8. $\frac{1}{10} + \frac{2}{100} + \frac{6}{1000} + \frac{24}{10{,}000} + \cdots$

9. $\frac{6}{5} + \frac{2}{5} + \frac{2}{15} + \cdots$

10. Finance Ms. Fuentes invests $500 quarterly (January 1, April 1, July 1, and October 1) in a retirement account that pays an APR of 12% compounded quarterly. Interest for each quarter is posted on the last day of the quarter. Determine the value of her investment at the end of the year. (Lesson 12-2)

Extra Practice See p. A49.

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