10-5

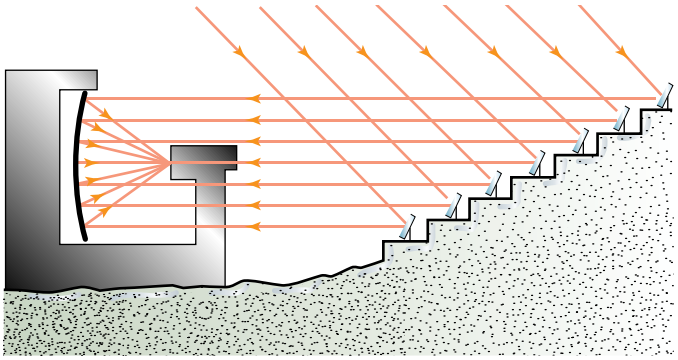
Parabolas

OBJECTIVES

- Use and determine the standard and general forms of the equation of a parabola.
- Graph parabolas.

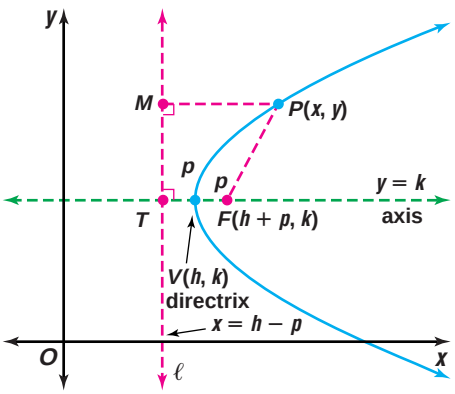
Real World Application

ENERGY The Odeillo Solar Furnace, located in southern France, uses a series of 63 flat mirrors, arranged on terraces on a hillside, to reflect the sun's rays on to a large parabolic mirror. These computer-controlled mirrors tilt to track the sun and ensure that its rays are always reflected to the central parabolic mirror.



This mirror in turn reflects the sun's rays to the focal point where a furnace is mounted on a tower. The concentrated energy generates temperatures of up to 6870°F. If the width of the Odeillo parabolic mirror is 138 feet and the furnace is located 58 feet from the center of the mirror, how deep is the mirror? *This problem will be solved in Example 2.*

In Chapter 4, you learned that the graphs of quadratic equations like $x = y^2$ or $y = x^2$ are called *parabolas.* A parabola is defined as the set of all points in a plane that are the same distance from a given point, called the **focus,** and a given line, called the **directrix.** *Remember that the distance from a point to a line is the length of the segment from the point perpendicular to the line.*



In the figure at the left, F is the focus of the parabola and ℓ is the directrix. This parabola is symmetric with respect to the line $y = k$, which passes through the focus. This line is called the **axis of symmetry,** or, more simply, the *axis* of the parabola. The point at which the axis intersects the parabola is called the **vertex.**

Suppose the vertex V has coordinates (h, k). Let p be the distance from the focus to the vertex, FV. By the definition of a parabola, the distance from any point on the parabola to the focus must equal the distance from that point to the directrix. So, if $FV = p$, then $VT = p$. The coordinates of F are $(h + p, k)$, and the equation of the directrix is $x = h - p$.

Now suppose that $P(x, y)$ is any point on the parabola other than the vertex. From the definition of a parabola, you know that $PF = PM$. Since M lies on the directrix, the coordinates of M are $(h - p, y)$.

For PF, let $F(h + p, k)$ be (x_1, y_1) and $P(x, y)$ be (x_2, y_2). Then for PM, let M be (x_1, y_1). You can use the Distance Formula to determine the equation for the parabola.

$$PF = PM$$

$$\sqrt{[x - (h + p)]^2 + (y - k)^2} = \sqrt{[x - (h - p)]^2 + (y - y)^2}$$

$$[x - (h + p)]^2 + (y - k)^2 = [x - (h - p)]^2 \quad \textit{Square each side.}$$

$$x^2 - 2x(h + p) + (h + p)^2 + (y - k)^2 = x^2 - 2x(h - p) + (h - p)^2$$

This equation can be simplified to obtain the equation

$$(y - k)^2 = 4p(x - h).$$

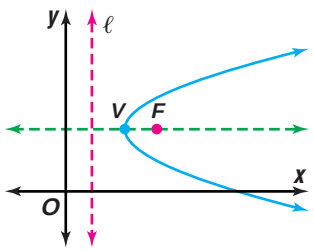
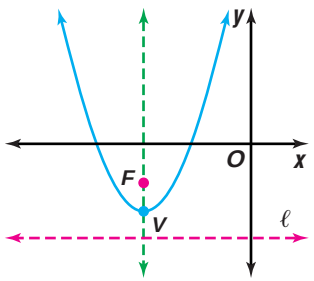
When p is positive, the parabola opens to the right.
When p is negative, the parabola opens to the left.

Unlike the equations of other conic sections, the equation of a parabola has only one squared term.

This is the equation of a parabola whose directrix is parallel to the y-axis. The equation of a parabola whose directrix is parallel to the x-axis can be obtained by switching the terms in the parentheses of the previous equation.

$$(x - h)^2 = 4p(y - k)$$

When p is positive, the parabola opens upward.
When p is negative, the parabola opens downward.

Standard Form of the Equation of a Parabola	Orientation when $p > 0$	Description
$(y - k)^2 = 4p(x - h)$		vertex: (h, k) focus: $(h + p, k)$ axis of symmetry: $y = k$ directrix: $x = h - p$ opening: right if $p > 0$ left if $p < 0$
$(x - h)^2 = 4p(y - k)$		vertex: (h, k) focus: $(h, k + p)$ axis of symmetry: $x = h$ directrix: $y = k - p$ opening: upward if $p > 0$ downward if $p < 0$

Example 1 **Consider the equation $y^2 = 8x + 48$.**

a. Find the coordinates of the focus and the vertex and the equations of the directrix and the axis of symmetry.

b. Graph the equation of the parabola.

a. First, write the equation in the form $(y - k)^2 = 4p(x - h)$.

$$\begin{aligned} y^2 &= 8x + 48 \\ y^2 &= 8(x + 6) && \textit{Factor.} \\ (y - 0)^2 &= 4(2)(x + 6) && \textit{4p = 8, so p = 2} \end{aligned}$$

In this form, we can see that $h = -6$, $k = 0$, and $p = 2$. We can use this to find the desired information.

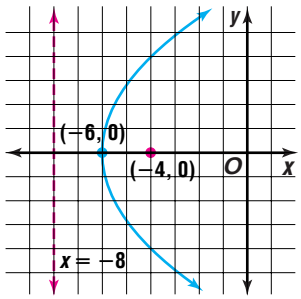
Vertex: $(-6, 0)$	(h, k)
Directrix: $x = -8$	$x = h - p$
Focus: $(-4, 0)$	$(h + p, k)$
Axis of Symmetry: $y = 0$	$y = k$

The axis of symmetry is the x-axis. Since p is positive, the parabola opens to the right.

Graphing Calculator Tip

You can graph a parabola that opens to the right or to left by first solving for y and then graphing the two resulting equations on the same screen.

b. Graph the directrix, the vertex, and the focus. To determine the shape of the parabola, graph several other ordered pairs that satisfy the equation and connect them with a smooth curve.



One useful property of parabolic mirrors is that all light rays traveling parallel to the mirror's axis of symmetry will be reflected by the parabola to the focus.

Example 2 **ENERGY** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Find and graph the equation of a parabola that models the shape of the Odeillo mirror.

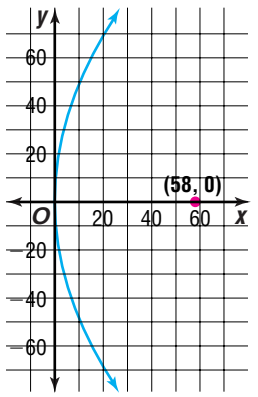
b. Find the depth of the parabolic mirror.

a. The shape of the mirror can be modeled by a parabola with vertex at the origin and opening to the right. The general equation of such a parabola is $y^2 = 4px$, where p is the focal length. Given a focal length of 58 feet, we can derive the model equation.

$$y^2 = 4px$$

$$y^2 = 4(58)x \quad p = 58$$

$$y^2 = 232x$$



b. With the mirror's vertex at the origin, the distance from the vertex to one edge of the mirror is half the overall width of the mirror, $\frac{1}{2}$(138 feet) or 69 feet.

Use the model equation to find the depth x of the mirror when the distance from the center is 69 feet.

$$\begin{aligned} y^2 &= 232x \\ (69)^2 &= 232x \quad y = 69 \\ 4761 &= 232x \\ x &= \frac{4761}{232} \text{ or about } 20.5 \end{aligned}$$

The mirror is about 20.5 feet deep.

You can use the same process you used with circles to rewrite the standard form of the equation of a parabola in general form. *You will derive the general form in Exercise 37.*

General Form for the Equation of a Parabola

The general form of the equation of a parabola is
$y^2 + Dx + Ey + F = 0$, when the directrix is parallel to the y-axis, or
$x^2 + Dx + Ey + F = 0$, when the directrix is parallel to the x-axis.

It is necessary to convert an equation in general form to standard form to determine the coordinates of the vertex (h, k) and the distance from the vertex to the focus p.

Example 3 **Consider the equation $2x^2 - 8x + y + 6 = 0$.**

a. Write the equation in standard form.

b. Find the coordinates of the vertex and focus and the equations for the directrix and the axis of symmetry.

c. Graph the equation of the parabola.

a. Since x is squared, the directrix of this parabola is parallel to the x-axis.

$$\begin{aligned} 2x^2 - 8x + y + 6 &= 0 \\ 2x^2 - 8x &= -y - 6 && \textit{Isolate the } x \textit{ terms and the } y \textit{ terms.} \\ 2(x^2 - 4x + ?) &= -y - 6 + ? && \textit{The GCF of the } x \textit{ terms is 2.} \\ 2(x^2 - 4x + 4) &= -y - 6 + 2(4) && \textit{Complete the square.} \\ 2(x - 2)^2 &= -(y - 2) && \textit{Simplify and factor.} \\ (x - 2)^2 &= -\frac{1}{2}(y - 2) && \textit{Divide each side by 2.} \end{aligned}$$

The standard form of the equation is $(x - 2)^2 = -\frac{1}{2}(y - 2)$.

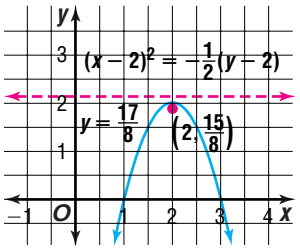
b. Since $4p = -\frac{1}{2}$, $p = -\frac{1}{8}$.

vertex: $(2, 2)$ (h, k) focus: $\left(2, \frac{15}{8}\right)$ $(h, k + p)$

directrix: $y = \frac{17}{8}$ $y = k - p$ axis of symmetry: $x = 2$ $x = h$

CONTENTS

c. Now sketch the graph of the parabola using the information found in part b.

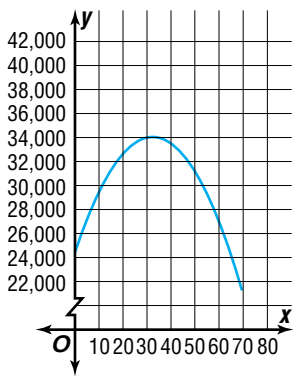


Parabolas are often used to demonstrate maximum or minimum points in real-world situations.

Example 4 **AERONAUTICS** **NASA's KC-135A aircraft flies in parabolic arcs to simulate the weightlessness experienced by astronauts in space. The aircraft starts its ascent at 24,000 feet. During the ascent, all on board experience 2g's or twice the pull of Earth's gravity. As the aircraft approaches its maximum height, the engines are stopped, and the aircraft is allowed to free fall at a precisely determined angle. Zero gravity is achieved for 25 seconds as the plane reaches the top of the parabola and begins its descent. After this 25-second period, the engines are throttled to bring the aircraft out of the dive. If the height of the aircraft in feet (y) versus time in seconds (x) is modeled by the equation $x^2 - 65x + 0.11y - 2683.75 = 0$, what is the maximum height achieved by the aircraft during its parabolic flight?**

First, write the equation in standard form.

$$x^2 - 65x + 0.11y - 2683.75 = 0$$
$$x^2 - 65x = -0.11y + 2683.75 \quad \textit{Isolate the } x \textit{ terms and } y \textit{ terms.}$$
$$x^2 - 65x + 1056.25 = -0.11y + 2683.75 + 1056.25 \quad \textit{Complete the square.}$$
$$(x - 32.5)^2 = -0.11y + 3740$$
$$(x - 32.5)^2 = -0.11(y - 34{,}000)$$

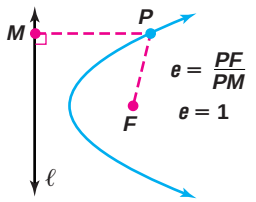
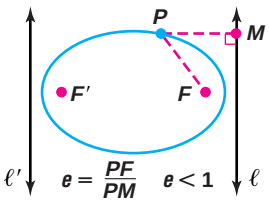
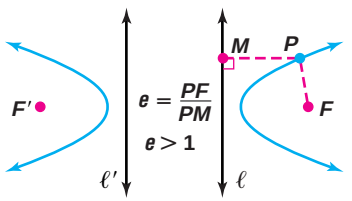


The vertex of the parabola is at (32.5, 34,000).

Remember that the vertex is the maximum or minimum point of a parabola. Since the parabola opens downward, the vertex is the maximum.

The x-coordinate of the vertex, 32.5, represents 32.5 seconds after the aircraft began the parabolic maneuver. The y-coordinate, 34,000, represents a maximum height of 34,000 feet.

All conics can be defined using the focus-directrix definition presented in this lesson. A conic section is defined to be the **locus** of points such that, for any point P in the locus, the ratio of the distance between that point and a fixed point F to the distance between that point and a fixed line ℓ, is constant. As we have seen, the point F is called the focus, and the line ℓ is the directrix. That ratio is the eccentricity of the curve, and its value can be used to determine the conic's classification. In the case of a parabola, $e = 1$. As shown previously, if $0 < e < 1$, the conic is an ellipse. If $e = 0$, the conic is a circle, and if $e > 1$, the conic is a hyperbola.

parabola	ellipse	hyperbola
$e = 1$	$e < 1, e \neq 0$	$e > 1$
		

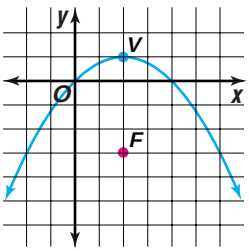
For those conics having more than one focus and directrix, F' and ℓ' represent alternates that define the same conic.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** a way in which you might distinguish the equation of a parabola from the equation of a hyperbola.
2. **Write** the equation of the graph shown at the right.
3. **Describe** the relationships among the vertex, focus, directrix and axis of symmetry of a parabola.
4. **Write** the equation in standard form of a parabola with vertex at $(-4, 5)$, opening to the left, and with a focus 5 units from its vertex.
5. **Identify** each of the following conic sections given their eccentricities.



a. $e = \frac{1}{2}$ b. $e = 1$ c. $e = 1.25$ d. $e = 0$

Guided Practice

For the equation of each parabola, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

6. $x^2 = 12(y - 1)$ 7. $y^2 - 4x + 2y + 5 = 0$ 8. $x^2 + 8x + 4y + 8 = 0$

Write the equation of the parabola that meets each set of conditions. Then graph the equation.

9. The vertex is at the origin, and the focus is at $(0, -4)$.
10. The parabola passes through the point at $(2, -1)$, has its vertex at $(-7, 5)$, and opens to the right.
11. The parabola passes through the point at $(5, 2)$, has a vertical axis, and has a minimum at $(4, -3)$.

12. **Sports** In 1998, Sammy Sosa of the Chicago Cubs was in a homerun race with Mark McGwire of the St. Louis Cardinals. One day, Mr. Sosa popped a baseball straight up at an initial velocity v_0 of 56 feet per second. Its distance s above the ground after t seconds is described by $s = v_0t - 16t^2 + 3$.
   a. Graph the function $s = v_0t - 16t^2 + 3$ for the given initial velocity.
   b. Find the maximum height achieved by the ball.
   c. If the ball is allowed to fall to the ground, how many seconds, to the nearest tenth, is it in the air?

EXERCISES

Practice

For the equation of each parabola, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

13. $y^2 = 8x$
14. $x^2 = -4(y - 3)$
15. $(y - 6)^2 = 4x$
16. $y^2 + 12x = 2y - 13$
17. $y - 2 = x^2 - 4x$
18. $x^2 + 10x + 25 = -8y + 24$
19. $y^2 - 2x + 14y = -41$
20. $y^2 - 2y - 12x + 13 = 0$
21. $2x^2 - 12y - 16x + 20 = 0$
22. $3x^2 - 30y - 18x + 87 = 0$
23. Consider the equation $2y^2 + 16y + 16x + 64 = 0$. Identify the coordinates of the vertex and focus and the equations of the directrix and axis of symmetry. Then graph the equation.

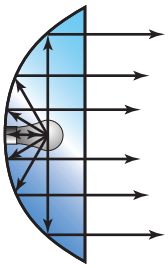
Write the equation of the parabola that meets each set of conditions. Then graph the equation.

24. The vertex is at $(-5, 1)$, and the focus is at $(2, 1)$.
25. The equation of the axis is $y = 6$, the focus is at $(0, 6)$, and $p = -3$.
26. The focus is at $(4, -1)$, and the equation of the directrix is $y = -5$.
27. The parabola passes through the point at $(5, 2)$, has a vertical axis, and has a maximum at $(4, 3)$.
28. The parabola passes through the point at $(-3, 1)$, has its vertex at $(-2, -3)$, and opens to the left.
29. The focus is at $(-1, 7)$, the length from the focus to the vertex is 2 units, and the function has a minimum.
30. The parabola has a vertical axis and passes through points at $(1, -7)$, $(5, -3)$, and $(7, -4)$.
31. The parabola has a horizontal axis and passes through the origin and points at $(-1, 2)$ and $(3, -2)$.
32. The parabola's directrix is parallel to the x-axis, and the parabola passes through points at $(1, 1)$, $(0, 9)$, and $(2, 1)$.

Applications and Problem Solving

Real World Application

33. Automotive Automobile headlights contain parabolic reflectors, which work on the principle that light placed at the focus of a parabola will reflect off the mirror-like surface in lines parallel to the axis of symmetry. Suppose a bulb is placed at the focus of a headlight's reflector, which is 2 inches from the vertex.



a. If the depth of the headlight is to be 4 inches, what should the diameter of the headlight be at its opening?

b. Find the diameter of the headlight at its opening if the depth is increased by 25%.

34. Business An airline has been charging \$140 per seat for a one-way flight. This flight has been averaging 110 passengers but can transport up to 180 passengers. The airline is considering a decrease in the price for a one-way ticket during the winter months. The airline estimates that for each \$10 decrease in the ticket price, they will gain approximately 20 passengers per flight.

a. Based on these estimates, what ticket price should the airline charge to achieve the greatest income on an average flight?

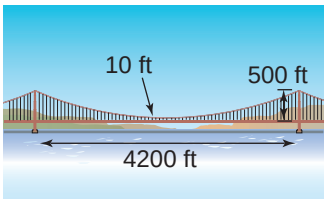
b. New estimates reveal that the increase in passengers per flight is closer to 10 for each \$10 decrease in the original ticket price. To maximize income, what should the new ticket price be?

35. Critical Thinking Consider the standard form of the equation of a parabola in which the vertex is known but the value of p is not known.

a. As $|p|$ becomes greater, what happens to the shape of the parabola?

b. As $|p|$ becomes smaller, what happens to the shape of the parabola?

36. Construction The Golden Gate Bridge in San Francisco, California, is a catenary suspension bridge, which is very similar in appearance to a parabola. The main span cables are suspended between two towers that are 4200 feet apart and 500 feet above the roadway. The cable extends 10 feet above the roadway midway between the two towers.

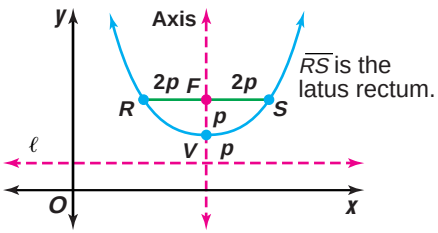


a. Find an equation that models the shape of the cable.

b. How far from the roadway is the cable 720 feet from the bridge's center?

37. Critical Thinking Using the standard form of the equation of a parabola, derive the general form of the equation of a parabola.

38. Critical Thinking The *latus rectum* of a parabola is the line segment through the focus that is perpendicular to the axis and has endpoints on the parabola. The length of the latus rectum is $|4p|$ units, where p is the distance from the vertex to the focus.



a. Write the equation of a parabola with vertex at $(-3, 2)$, axis $y = 2$, and latus rectum 8 units long.

b. The latus rectum of the parabola with equation $(x - 1)^2 = -16(y - 4)$ coincides with the diameter of a circle. Write the equation of the circle.

Mixed Review

39. Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{16} = 1$. Then graph the equation. *(Lesson 10-4)*

40. Find the coordinates of the center, foci, and vertices of the ellipse whose equation is $4x^2 + 25y^2 + 250y + 525 = 0$. Then graph the ellipse. *(Lesson 10-3)*

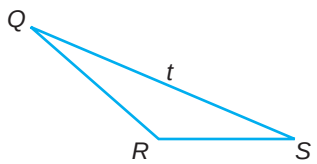
41. Graph $r = 12 \cos 2\theta$. *(Lesson 9-2)*

42. Find the values of θ for which $\cos \theta = 1$ is true. *(Lesson 6-3)*

43. Geometry A regular hexagon is inscribed in a circle with a radius 6.4 centimeters long. Find the apothem; that is, the distance from the center of the circle to the midpoint of a side. *(Lesson 5-4)*

44. Describe the end behavior of $g(x) = \frac{4}{x^2 + 1}$. *(Lesson 3-5)*

45. SAT/ACT Practice Triangle QRS has sides of lengths 14, 19, and t, where t is the length of the longest side. If t is the cube of an integer, what is the perimeter of the triangle?



A 41 **B** 58 **C** 60 **D** 69 **E** 76

MID-CHAPTER QUIZ

1. Given: $A(3, 3)$, $B(6, 9)$, and $C(9, 3)$ (Lesson 10-1)
- **a.** Show that these points form an isosceles triangle.
- **b.** Determine the perimeter of the triangle to the nearest hundredth.

2. Determine the midpoint of the diagonals of the rectangle with vertices $A(-4, 9)$, $B(5, 9)$, $C(5, 5)$, and $D(-4, 5)$. (Lesson 10-1)

3. Find the coordinates of the center and radius of the circle with equation $x^2 + y^2 - 6y - 8x = -16$. Then graph the circle. (Lesson 10-2)

4. Write the equation of the circle with center at $(-5, 2)$ and radius $\sqrt{7}$. (Lesson 10-2)

5. Astronomy A satellite orbiting Earth follows an elliptical path with the center of Earth as one focus. The eccentricity of the orbit is 0.16, and the major axis is 10,440 miles long. (Lesson 10-3)
- **a.** If the mean diameter of Earth is 7920 miles, find the greatest and least distance of the satellite from the surface of Earth.
- **b.** Assuming that the center of the ellipse is the origin and the foci lie on the x-axis, write the equation of the orbit of the satellite.

6. Identify the center, vertices, and foci of the ellipse with equation $9x^2 + 25y^2 - 72x + 250y + 544 = 0$. Then graph the equation. (Lesson 10-3)

7. Identify the center, vertices, foci, and equations of the asymptotes of the graph of the hyperbola with equation $3y^2 + 24y - x^2 - 2x + 41 = 0$. Then graph the equation. (Lesson 10-4)

8. Write the equation of a hyperbola that passes through the point at (4, 2) and has asymptotes with equations $y = 2x$ and $y = -2x + 4$. (Lesson 10-4)

9. Identify the vertex, focus, and equations of the axis of symmetry and directrix for the parabola with equation $y^2 - 4x + 2y + 5 = 0$. Then graph the equation. (Lesson 10-5)

10. Write the equation of the parabola that passes through the point at $(9, -2)$, has its vertex at $(5, -1)$, and opens downward. (Lesson 10-5)

10-6

Rectangular and Parametric Forms of Conic Sections

OBJECTIVES

- Recognize conic sections in their rectangular form by their equations.
- Find a rectangular equation for a curve defined parametrically and vice versa.

Real World Application

TRANSPORTATION The first self-propelled boats on the western rivers of the United States were the paddlewheels. This boat used a steam engine to turn one or more circular wheels that had a paddle attached to the end of each spoke. In 1811, Robert Fulton and Nicholas Roosevelt built the first paddlewheel large enough for commercial use on the Ohio and Mississippi Rivers. By the end of the 19th century, paddlewheel boats had fought wars and carried people and cargo on nearly every river in the United States. Despite advancements in technology, paddlewheels are still in use today, though mainly for sentimental reasons. *You will solve a problem related to this in Exercise 40.*

We have determined a general equation for each conic section we have studied. All of these equations are forms of the general equation for conic sections.

General Equation for Conic Sections

The equation of a conic section can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not all zero.

The graph of a second-degree equation in two variables always represents a conic or degenerate case, unless the equation has no graph at all in the real number plane. Most of the conic sections that we have studied have axes that are parallel to the coordinate axes. The general equations of these conics have no xy term; thus, $B = 0$. The one conic section we have discussed whose axes are not parallel to the coordinate axes is the hyperbola whose equation is $xy = k$. In its equation, $B \neq 0$.

To identify the conic section represented by a given equation, it is helpful to write the equation in standard form. However, when $B = 0$, you can also identify the conic section by how the equation compares to the general equation. The table on the next page summarizes the standard forms and differences among the general forms.

The circle is actually a special form of the ellipse, where $a^2 = b^2 = r^2$.

General Form: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$		
Conic Section	**Standard Form of Equation**	**Variation of General Form of Conic Equations**
circle	$(x - h)^2 + (y - k)^2 = r^2$	$A = C$
parabola	$(y - k)^2 = 4p(x - h)$ or $(x - h)^2 = 4p(y - k)$	Either A or C is zero.
ellipse	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ or $\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$	A and C have the same sign and $A \neq C$.
hyperbola	$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ or $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$	A and C have opposite signs.
	$xy = k$	$A = C = D = E = 0$

Remember that graphs can also be degenerate cases.

Example 1 **Identify the conic section represented by each equation.**

a. $6y^2 + 3x - 4y - 12 = 0$

$A = 0$ and $C = 6$. Since $A = 0$, the conic is a parabola.

b. $3y^2 - 2x^2 + 5y - x - 15 = 0$

$A = -2$ and $C = 3$. Since A and C have different signs, the conic is a hyperbola.

c. $9x^2 + 27y^2 - 6x - 108y + 82 = 0$

$A = 9$ and $C = 27$. Since A and C have the same signs and are not equal, the conic is an ellipse.

d. $4x^2 + 4y^2 + 5x + 2y - 150 = 0$

$A = 4$ and $C = 4$. Since $A = C$, the conic is a circle.

Look Back
You can refer to Lesson 8-6 to review writing and graphing parametric equations.

So far we have discussed equations of conic sections in their rectangular form. Some conic sections can also be described parametrically.

The general form for a set of parametric equations is

$$x = f(t) \text{ and } y = g(t), \text{ where } t \text{ is in some interval } I.$$

As t varies over I in some order, a curve containing points (x, y) is traced out in a certain direction.

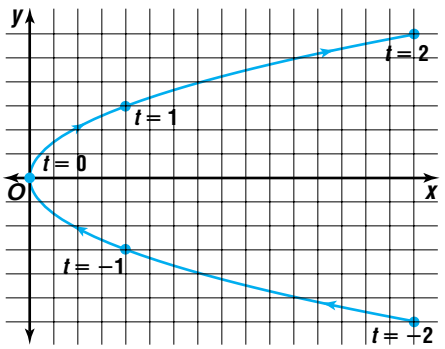
A parametric equation can be transformed into its more familiar rectangular form by eliminating the parameter t from the parametric equations.

Example 2 **Graph the curve defined by the parametric equations $x = 4t^2$ and $y = 3t$, where $-2 \le t \le 2$. Then identify the curve by finding the corresponding rectangular equation.**

Make a table of values assigning values for t and evaluating each expression to find values for x and y.

t	x	y	(x, y)
-2	16	-6	$(16, -6)$
-1	4	-3	$(4, -3)$
0	0	0	$(0, 0)$
1	4	3	$(4, 3)$
2	16	6	$(16, 6)$

Then graph the curve.



Notice the arrows indicating the direction in which the curve is traced for increasing values of t.

The graph appears to be part of a parabola. To identify the curve accurately, find the corresponding rectangular equation by eliminating t from the given parametric equations.

First, solve the equation $y = 3t$ for t.

$$y = 3t$$
$$\frac{y}{3} = t \quad \textit{Solve for t.}$$

Then substitute $\frac{y}{3}$ for t in the equation $x = 4t^2$.

$$x = 4t^2$$
$$x = 4\left(\frac{y}{3}\right)^2 \quad t = \frac{y}{3}$$
$$x = \frac{4y^2}{9}$$

The equation $x = \frac{4y^2}{9}$ is the equation of a parabola with vertex at $(0, 0)$ and its axis of symmetry along the x-axis. Notice that the domain of the rectangular equation is $x \ge 0$, which is greater than that of its parametric representation. By restricting the domain to $0 \le x \le 16$, our rectangular representation matches our parametric representation for the graph.

Some parametric equations require the use of trigonometric identities to eliminate the parameter t.

Example 3 **Find the rectangular equation of the curve whose parametric equations are $x = 2 \cos t$ and $y = 2 \sin t$, where $0 \le t \le 2\pi$. Then graph the equation using arrows to indicate how the graph is traced.**

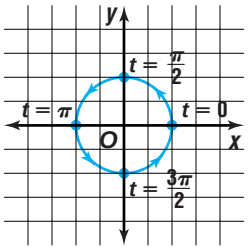
Solve the first equation for $\cos t$ and the second equation for $\sin t$.

$$\cos t = \frac{x}{2} \text{ and } \sin t = \frac{y}{2}$$

Use the trigonometric identity $\cos^2 t + \sin^2 t = 1$ to rewrite the equation to eliminate t.

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \textit{Substitution}$$
$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$
$$x^2 + y^2 = 4 \quad \textit{Multiply each side by 4.}$$

This is the equation of a circle with center at (0, 0) and radius 2. As t increases from $t = 0$ to $t = 2\pi$, we see that the curve is traced in a counterclockwise motion.



Graphing Calculator Appendix

For keystroke instructions on how to graph parametric equations, see page A21.

You can also use substitution to find the parametric equations for a given conic section. If the conic section is defined as a function, $y = f(x)$, one way of finding the parametric equations is by letting $x = t$ and $y = f(t)$, where t is in the domain of f.

Example 4 **Find parametric equations for the equation $y = x^2 + 3$.**

Let $x = t$. Then $y = t^2 + 3$. Since the domain of the function $f(t)$ is all real numbers, the parametric equations are $x = t$ and $y = t^2 + 3$, where $-\infty < t < \infty$.

GRAPHING CALCULATOR EXPLORATION

The graph of the parametric equations $x = \cos t$ and $y = \sin t$, where $0 \le t \le 2\pi$ is the unit circle. Interchanging the trigonometric functions or changing the coefficients can alter the graph's size and shape as well as its starting point and the direction in which it is traced. Watch while the graph is being drawn to see the effects.

TRY THESE

1. Graph the parametric equations $x = -\cos t$ and $y = \sin t$, where $0 \le t \le 2\pi$.
   a. Where does the graph start?
   b. In which direction is the graph traced?

2. Graph the parametric equations $x = \sin t$ and $y = \cos t$, where $0 \le t \le 2\pi$.
   a. Where does the graph start?
   b. In which direction is the graph traced?

3. Graph $x = 2 \cos t$, and $y = 3 \sin t$, where $0 \le t \le 2\pi$. What is the shape of the graph?

WHAT DO YOU THINK?

4. What is the significance of the number a in the equations $x = a \cos t$ and $y = a \sin t$, where $0 \le t \le 2\pi$?

5. What is the result of changing the interval to $0 \le t \le 4\pi$ in Exercises 1-3?

Parametric equations are particularly useful in describing the motion of an object along a curved path.

Example 5 **ASTRONOMY** **The orbit of Saturn around the sun is modeled by the equation $\frac{x^2}{(9.50)^2} + \frac{y^2}{(9.48)^2} = 1$. It takes Saturn approximately 30 Earth years to complete one revolution of its orbit.**

a. Find parametric equations that model the motion of Saturn beginning at (9.50, 0) and moving counterclockwise around the sun.

b. Use the parametric equations to determine Saturn's position after 18 years.

a. From the given equation, you can determine that the orbital path of Saturn is an ellipse with a major axis of 9.50 AU and a minor axis of 9.48 AU.

Like a circle, the parametric representation for an ellipse involves the use of sines and cosines. The parametric representation for the given equation is an ellipse with $x = 9.5$ and $y = 0$ when $t = 0$, so the following equations are true.

$$\frac{x}{9.50} = \cos \omega t \qquad \text{and} \qquad \frac{y}{9.48} = \sin \omega t$$

You can verify that by using the equations $x = 9.50$ and $y = 0$ when $t = 0$ and $\cos^2 \omega t + \sin^2 \omega t = 1$.

To move counterclockwise, the motion will have to begin with the value of x decreasing and y increasing, so $\omega > 0$. Since Saturn completes an orbit in 30 Earth years, the sine and cosine functions have a period $\frac{2\pi}{\omega} = 30$, so $\omega = \frac{\pi}{15}$.

Thus, the parametric equations corresponding to the rectangular equation $\frac{x^2}{(9.50)^2} + \frac{y^2}{(9.48)^2} = 1$ are $x = 9.50 \cos \frac{\pi}{15}t$ and $y = 9.48 \sin \frac{\pi}{15}t$, where $0 \le t \le 30$.

You can verify the equations above using a graphing calculator to trace the ellipse.

b. The position of Saturn after 18 years is found by letting $t = 18$ in both parametric equations.

$$x = 9.50 \cos\left(\frac{\pi}{15}t\right)$$

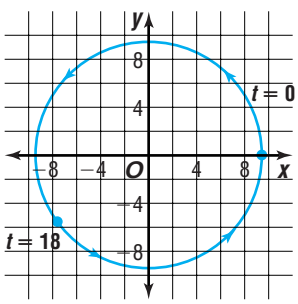
$$x = 9.50 \cos\left[\frac{\pi}{15}(18)\right] \qquad t = 18$$

$$x \approx -7.69$$

$$y = 9.48 \sin\left(\frac{\pi}{15}t\right)$$

$$y = 9.48 \sin\left[\frac{\pi}{15}(18)\right] \qquad t = 18$$

$$y \approx -5.57$$



Eighteen years later, Saturn is located at $(-7.69, -5.57)$.

CONTENTS

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** the general form of the equations of the four conic sections we have studied.
2. **Give** new restrictions on the parameter t in Example 2 so that the domains of the rectangular and parametric equations are the same.
3. **Write** the rectangular equation of a parabola with vertex at the origin and opening to the left. Then write the parametric equations that correspond to that parabola.

Guided Practice

Identify the conic section represented by each equation. Then write the equation in standard form and graph the equation.

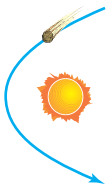
4. $x^2 + 9y^2 + 2x - 18y + 1 = 0$
5. $y^2 - 8x = -8$
6. $x^2 - 4x - y^2 - 5 - 4y = 0$
7. $x^2 - 6x + y^2 - 12y + 41 = 0$

Find the rectangular equation of the curve whose parametric equations are given. Then graph the equation using arrows to indicate orientation.

8. $x = t, y = -t^2 - 6t + 2; -\infty < t < \infty$
9. $x = 2 \cos t, y = 3 \sin t; 0 \le t \le 2\pi$

Find parametric equations for each rectangular equation.

10. $y = 2x^2 - 5x$
11. $x^2 + y^2 = 36$



12. **Astronomy** Some comets traveling at great speeds follow parabolic paths with the sun as their focus. Suppose the motion of a certain comet is modeled by the parametric equations $x = \frac{t^2}{80}$, and $y = t$ for $-\infty < t < \infty$. Find the rectangular equation that models the comet's path.

Exercises

Practice

Identify the conic section represented by each equation. Then write the equation in standard form and graph the equation.

13. $x^2 - 4y - 6x + 9 = 0$
14. $x^2 - 8x + y^2 + 6y + 24 = 0$
15. $x^2 - 3y^2 + 2x - 24y - 41 = 0$
16. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
17. $x^2 = y + 8x - 16$
18. $2xy = 3$
19. $5x^2 + 2y^2 - 40x - 20y + 110 = 0$
20. $x^2 - 8x + 11 = -y^2$
21. $8y^2 - 9x^2 - 16y + 36x - 100 = 0$
22. $4y^2 + 4y + 8x = 15$
23. Identify the conic section represented by $-4y^2 + 10x = 16y - x^2 - 5$. Write the equation in standard form and then graph the equation.
24. In the general equation of a conic, $A = C = 2, B = 0, D = -8, E = 12$, and $F = 6$. Write the equation in standard form. Then graph the equation.

Find the rectangular equation of the curve whose parametric equations are given. Then graph the equation using arrows to indicate orientation.

25. $x = t, y = 2t^2 - 4t + 1; -\infty < t < \infty$
26. $x = \cos 2t, y = \sin 2t; 0 \le t \le 2\pi$
27. $x = -\cos t, y = \sin t; 0 \le t \le 2\pi$
28. $x = 3 \sin t, y = 2 \cos t; 0 \le t \le 2\pi$
29. $x = -\sin 2t, y = 2 \cos 2t; 0 \le t \le \pi$
30. $x = 2t - 1, y = \sqrt{t}; 0 \le t \le 4$

31. Find a rectangular equation for the curve whose parametric equations are $x = -3 \cos 2t$ and $y = 3 \sin 2t$, $0 \leq t \leq 2\pi$.

Find parametric equations for each rectangular equation.

32. $x^2 + y^2 = 25$ **33.** $x^2 + y^2 - 16 = 0$ **34.** $\frac{x^2}{4} + \frac{y^2}{25} = 1$

35. $\frac{y^2}{16} + x^2 = 1$ **36.** $y = x^2 - 4x + 7$ **37.** $x = y^2 + 2y - 1$

38. Find parametric equations for the rectangular equation $(y + 3)^2 = 4(x - 2)$.

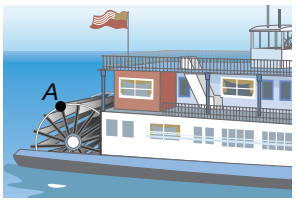
Graphing Calculator

39. Consider the rectangular equation $x = \sqrt{y}$.

- **a.** By using different choices for t, find two different parametric representations of this equation.
- **b.** Graph the rectangular equation by hand. Then use a graphing calculator to sketch the graphs of each set of parametric equations.
- **c.** Are your graphs from part **b** the same?
- **d.** What does this suggest about parametric representations of rectangular equations?

Applications and Problem Solving

40. Transportation A riverboat's paddlewheel has a diameter of 12 feet and at full speed, makes one clockwise revolution in 2 seconds.



- **a.** Write a rectangular equation to model the shape of the paddlewheel.
- **b.** Write parametric equations describing the position of a point A on the paddlewheel for any given time t. Assume that at $t = 0$, A is at the very top of the wheel.
- **c.** How far will point A, which is a fixed point on the wheel, move in 1 minute?

41. Critical Thinking Identify the graph of each equation using the method described in this lesson. Then identify the graph of each equation after first rewriting the equation in standard form and solving for y. Explain the discrepancies, if any, in your answers.

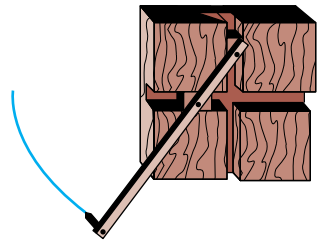
a. $2x^2 + 5y^2 = 0$ **b.** $x^2 + y^2 - 4x - 6y + 13 = 0$ **c.** $y^2 - 9x^2 = 0$

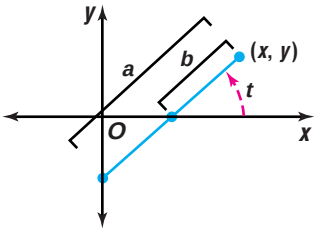
42. Critical Thinking Explain why a substitution of $x = t^2$ is not appropriate when trying to find a parametric representation of $y = x^2 - 5$?

43. Timing The path traced by the tip of the second-hand of a clock can be modeled by the equation of a circle in parametric form.

- **a.** If the radius of the clock is 6 inches, find an equation in rectangular form that models the shape of the clock.
- **b.** Find parametric equations that describe the motion of the tip as it moves from 12 o'clock noon to 12 o'clock noon of the next day.
- **c.** Simulate the motion described by graphing the equations on a graphing calculator.

44. Framing Portraits are often framed so that the opening through which the picture is seen is an ellipse. These oval mats must be custom cut using an oval cutter whose design relies upon the parametric equations of an ellipse. The elliptical compass at the right consists of a stick with a pencil attached to one end and two pivot holes at the other. Through these holes, the stick is anchored to two small blocks, one of which can slide horizontally and the other vertically in its groove. Use the diagram of the elliptical compass at right to verify that $x = a \cos t$ and $y = b \sin t$. (*Hint*: Draw an extra vertical and an extra horizontal line to create right triangles and then use trigonometry.)





Mixed Review

45. Find the coordinates of the vertex, focus, and the equations of the axis of symmetry and directrix of the parabola with equation $x^2 - 12y + 10x = -25$. Then graph the equation. *(Lesson 10-5).*

46. Graph $xy = -25$. *(Lesson 10-4)*

47. Write $3x^2 + 3y^2 - 18x + 12y = 9$ in standard form. Then graph the equation. *(Lesson 10-2)*

48. A 30-pound force is applied to an object at an angle of $60°$ with the horizontal. Find the magnitude of the horizontal and vertical components of the force. *(Lesson 8-1)*

49. Statistics The prediction equation $y = -0.13x + 37.8$ gives the fuel economy y for a car with horsepower x. Is the equation a better predictor for Car 1, which has a horsepower of 135 and average 19 miles per gallon, or for Car 2, which has a horsepower of 245 and averages 16 miles per gallon? Explain. *(Lesson 7-7)*

50. Find the value of $\sin\left(2 \operatorname{Sin}^{-1} \frac{1}{2}\right)$. *(Lesson 6-8)*

51. Find the area to the nearest square unit of $\triangle ABC$ if $a = 48$, $b = 32$, and $c = 44$. *(Lesson 5-8)*

52. Solve $\sqrt{2y - 3} - \sqrt{2y + 3} = -1$. *(Lesson 4-7)*

53. If y varies jointly as x and z and $y = 16$ when $x = 5$ and $z = 2$, find y when $x = 8$ and $z = 3$. *(Lesson 3-8)*

54. Find the determinant of $\begin{bmatrix} 5 & 9 \\ 7 & -3 \end{bmatrix}$. Then state whether an inverse exists for the matrix. *(Lesson 2-5)*

55. Write the point-slope form of the equation of the line through the points $(-6, 4)$ and $(3, 7)$. Then write the equation in slope-intercept form. *(Lesson 1-4)*

56. SAT/ACT Practice For all values where $x \neq y$, let $x \# y$ represent the lesser of the numbers x and y, and let $x @ y$ represent the greater of the number x and y. What is the value of $(1 \# 4) @ (2 \# 3)$?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Extra Practice See p. A45.

10-7 Transformations of Conics

OBJECTIVES
- Find the equations of conic sections that have been translated or rotated.
- Graph rotations and/or translations of conic equations.
- Identify the equations of conic sections using the discriminant.
- Find the angle of rotation for a given equation.

Real World Application

SPORTS At the 1996 Olympics in Atlanta, Georgia, U.S. high school student Kim Rhode won the gold medal in the women's double trap shooting event, being staged at the Olympics for the first time. The double trap event consists of firing double barrel shotguns at flying clay targets that are launched two at a time out of a house located 14.6 to 24.7 meters in front of the contestants. Targets are thrown out of the house in a random arc, but always at the same height. *A problem related to this will be solved in Example 1.*

Thus far, we have used a transformation called a translation to show how the parent graph of each of the conic sections is translated to a center other than the origin. For example, the equation of the circle $x^2 + y^2 = r^2$ becomes $(x - h)^2 + (y - k)^2 = r^2$ for a center of (h, k). A translation of a set of points with respect to (h, k) is often written as follows.

$T_{(h, k)}$ ⟷ translation with respect to (h, k)

Example 1 Real World Application

SPORTS **Refer to the application above. A video game simulating the sport of double trap allows a player to shoot at two elliptically-shaped targets released from a house at the bottom of the screen. With the house located at the origin, a target at its initial location is modeled by the equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Suppose a player misses one of the two targets released and the center of the target leaves the screen at the point (24, 30). Find an equation that models the shape and position of the target with its center translated to this point.**

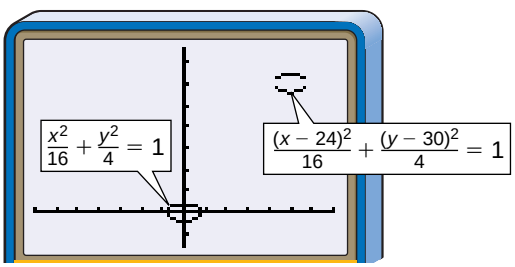
To write the equation of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ for $T_{(24, 30)}$, let $h = 24$ and $k = 30$. Then replace x with $x - h$ and y with $y - k$.

$x^2 \Rightarrow (x - 24)^2$

$y^2 \Rightarrow (y - 30)^2$

Thus, the translated equation is $\frac{(x - 24)^2}{16} + \frac{(y - 30)^2}{4} = 1$

The graph shows the parent ellipse and its translation.



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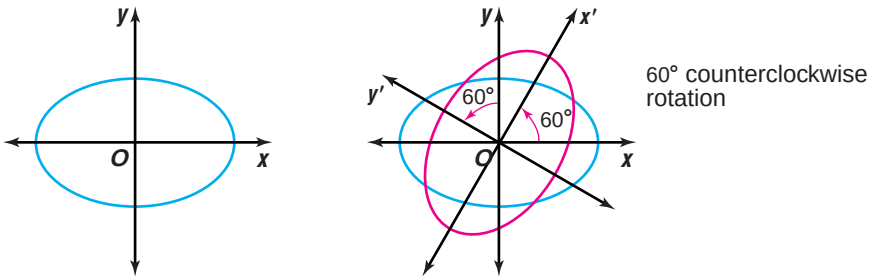
CONTENTS

Look Back
You can refer to Lesson 2-4 to review rotation.

Another type of transformation you have studied is a rotation. Except for hyperbolas whose equations are of the form $xy = k$, all of the conic sections we have studied thus far have been oriented with their axes parallel to the coordinate axes. In the general form of these conics $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $B = 0$. Whenever $B \neq 0$, then the axes of the conic section are not parallel to the coordinate axes. That is, the graph is rotated.

The figures below show an ellipse whose center is the origin and its rotation. Notice that the angle of rotation has the same measure as the angles formed by the positive x-axis and the major axis and the positive y-axis and the minor axis.



60° counterclockwise rotation

The coordinates of the points of a rotated figure can be found by using a rotation matrix.

A positive value of θ indicates a counterclockwise rotation. A negative value of θ indicates a clockwise rotation.

A rotation of θ about the origin can be described by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let $P(x, y)$ be a point on the graph of a conic section. Then let $P'(x', y')$ be the image of P after a counterclockwise rotation of θ. The values of x' and y' can be found by matrix multiplication.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

The inverse of the rotation matrix represents a rotation of $-\theta$. Multiply each side of the equation by the inverse rotation matrix to solve for x and y.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'\cos\theta + y'\sin\theta \\ -x'\sin\theta + y'\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'\cos\theta + y'\sin\theta \\ -x'\sin\theta + y'\cos\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The result is two equations that can be used to determine the equation of a conic with respect to a rotation of θ.

Rotation Equations

To find the equation of a conic section with respect to a rotation of θ, replace

x with $x'\cos\theta + y'\sin\theta$
and y with $-x'\sin\theta + y'\cos\theta$.

CONTENTS

Example 2 **Find the equation of the graph of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ after it is rotated 45° about the origin. Then sketch the graph and its rotation.**

The graph of this equation is a hyperbola.

Find the expressions to replace x and y.

Replace x with $x' \cos 45° + y' \sin 45°$ or $\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$.

Replace y with $-x' \sin 45° + y' \cos 45°$ or $-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$.

Computation is often easier if the equation is rewritten as an equation with denominators of 1.

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$9x^2 - 16y^2 = 144 \qquad \textit{Multiply each side by 144.}$$

$$9\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 - 16\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 144 \qquad \textit{Replace } x \textit{ and } y.$$

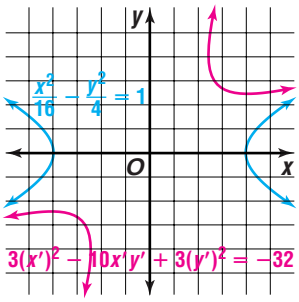
$$9\left[\frac{1}{2}(x')^2 + x'y' + \frac{1}{2}(y')^2\right] - 16\left[\frac{1}{2}(x')^2 - x'y' + \frac{1}{2}(y')^2\right] = 144 \qquad \textit{Expand the binomial.}$$

$$-\frac{7}{2}(x')^2 + 25x'y' - \frac{7}{2}(y')^2 = 144 \qquad \textit{Simplify.}$$

$$7(x')^2 - 50x'y' + 7(y')^2 = -288 \qquad \textit{Multiply each side by } -2.$$

The equation of the hyperbola after the 45° rotation is $7(x')^2 - 50x'y' + 7(y')^2 = -288$.

The graph below shows the hyperbola and its rotation.



In Lesson 10-6, you learned to identify a conic from its general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$. When $B \neq 0$, the equation can be identified by examining the discriminant of the equation.

You may remember from the Quadratic Formula that the discriminant of a second-degree equation is defined as $B^2 - 4AC$ and will remain unchanged under any rotation. That is, $B^2 - 4AC = (B')^2 - 4A'C'$.

Identifying Conics By Using the Discriminant

For the general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,
- if $B^2 - 4AC < 0$, the graph is a circle ($A = C$, $B = 0$) or an ellipse ($A \neq C$ or $B \neq 0$);
- if $B^2 - 4AC > 0$, the graph is a hyperbola;
- if $B^2 - 4AC = 0$, the graph is a parabola.

Remember that the graphs can also be degenerate cases.

Example 3 **Identify the graph of the equation $x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$.**

Since the equation contains an xy-term, use the discriminant of the equation to identify the conic.

$B^2 - 4AC = (-4)^2 - 4(1)(4)$ *$A = 1, B = -4, C = 4$*
$= 0$

Since $B^2 - 4AC = 0$, the graph of the equation is a parabola.

You can also use values from the general form to find the angle of rotation about the origin.

Angle of Rotation About the Origin

For the general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the angle of rotation θ about the origin can be found by

$$\theta = \frac{\pi}{4}, \text{ if } A = C, \text{ or}$$
$$\tan 2\theta = \frac{B}{A - C}, \text{ if } A \neq C.$$

Example 4 **Identify the graph of the equation $2x^2 + 9xy + 14y^2 - 5 = 0$. Then find θ and use a graphing calculator to draw the graph.**

$B^2 - 4AC = (9)^2 - 4(2)(14)$ *$A = 2, B = 9,$ and $C = 14$*
$= -31$

Since the discriminant is less than 0 and $A \neq C$, the graph is an ellipse.

Now find θ using $\tan 2\theta = \frac{B}{A - C}$, since $A \neq C$.

$$\tan 2\theta = \frac{B}{A - C}$$
$$\tan 2\theta = \frac{9}{2 - 14}$$
$$\tan 2\theta = -0.75$$
$$2\theta = -36.86989765 \quad \textit{Take the inverse tangent of each side.}$$
$$\theta \approx -18^\circ \quad \textit{Round to the nearest degree.}$$

To graph the equation, you must solve for y. Rewrite the equation in quadratic form, $ay^2 + by + c = 0$.

a b c
↓ ↓ ↓
$14y^2 + (9x)y + (2x^2 - 5) = 0$

(continued on the next page)

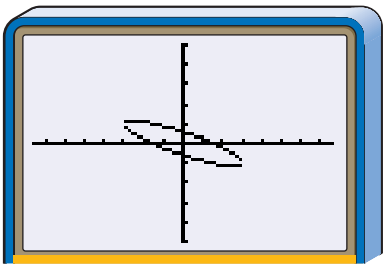
Now use the Quadratic Formula to solve for y.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(9x) \pm \sqrt{(9x)^2 - 4(14)(2x^2 - 5)}}{2(14)}$$

$$y = \frac{-9x \pm \sqrt{-31x^2 + 280}}{28}$$

Enter the equations and graph.



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This method of solving for y to graph a rotated equation is useful when trying to identify the equation of a degenerate conic.

Example 5 **The graph of $xy - y^2 + 2x^2 = 0$ is a degenerate case. Identify the graph and then draw it.**

In this equation, $B^2 - 4AC > 0$. At first glance, this equation may appear to be the equation of a hyperbola. A closer inspection reveals that this is a degenerate case.

Solve the equation for y by first rewriting the equation in quadratic form.

$$\underset{\downarrow}{a} \quad \underset{\downarrow}{b} \quad \underset{\downarrow}{c}$$

$$(-1)y^2 + (x)y + 2x^2 = 0$$

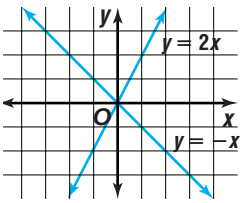
Now use the quadratic formula to solve for y.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-x \pm \sqrt{x^2 - 4(-1)(2x^2)}}{2(-1)}$$

$y = 2x$ or $y = -x$

So the graph of $xy - y^2 + 2x^2$ is actually the graph of $y = 2x$ and $y = -x$, which are intersecting lines.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Sketch** a parabola with vertex at the origin and a horizontal line of symmetry. Then sketch the parabola for $T_{(-3, 3)}$. Label each graph with its equation in standard form.
2. **Write** the expressions needed to replace x and y for an equation rotated 30°.
3. **Indicate** the angle of rotation about the origin needed to transform the equation $\frac{x^2}{100} + \frac{y^2}{25} = 1$ into $\frac{x^2}{25} + \frac{y^2}{100} = 1$.

4. **You Decide** Ebony says that $7x^2 - 6\sqrt{3}xy + 13y^2 = 0$ is the equation of an ellipse. Teisha disagrees. She says the equation is a hyperbola. Who is correct? Justify your answer.

Guided Practice **Identify the graph of each equation. Write an equation of the translated or rotated graph in general form.**

5. $x^2 + y^2 = 7$ for $T_{(3, 2)}$
6. $y = 2x^2 - 7x + 5$ for $T_{(-4, 5)}$
7. $x^2 - y^2 = 9, \theta = 60°$
8. $x^2 - 5x + y^2 = 3, \theta = \frac{\pi}{4}$

Identify the graph of each equation. Then find θ to the nearest degree.

9. $9x^2 + 4xy + 4y^2 + 2 = 0$
10. $8x^2 + 5xy - 4y^2 + 2 = 0$

11. The graph of $3(x - 1)^2 + 4(y + 4)^2 = 0$ is a degenerate case. Identify the graph and then draw it.

12. **Communications** A satellite dish tracks a satellite directly overhead. Suppose the equation $y = \frac{1}{6}x^2$ models the shape of the dish when it is oriented in this position. Later in the day the dish is observed to have rotated approximately 30°.
    a. Find an equation that models the new orientation of the dish.
    b. Sketch the graphs of both equations on the same set of axes using a graphing calculator.

EXERCISES

Practice **Identify the graph of each equation. Write an equation of the translated or rotated graph in general form.**

13. $y = 3x^2 - 2x + 5$ for $T_{(2, -3)}$
14. $4x^2 + 5y^2 = 20$ for $T_{(5, -6)}$
15. $3x^2 + y^2 = 9$ for $T_{(-1, 3)}$
16. $4y^2 + 12x^2 = 24$ for $T_{(-1, 4)}$
17. $9x^2 - 25y^2 = 225$ for $T_{(0, 5)}$
18. $(x + 3)^2 = 4y$ for $T_{(-7, 2)}$
19. $x^2 - 8y = 0, \theta = 90°$
20. $2x^2 + 2y^2 = 8, \theta = 30°$
21. $y^2 + 8x = 0, \theta = \frac{\pi}{6}$
22. $xy = -8, \theta = \frac{\pi}{4}$
23. $x^2 - 5x + y^2 = 3, \theta = \frac{\pi}{3}$
24. $16x^2 - 4y^2 = 64, \theta = 60°$

25. Write the equation of the ellipse $6x^2 + 5y^2 = 30$ after a rotation of 30° about the origin.

Identify the graph of each equation. Then find θ to the nearest degree.

26. $9x^2 + 4xy + 5y^2 - 40 = 0$
27. $x^2 - xy - 4y^2 - x - y + 4 = 0$
28. $8x^2 + 8xy + 2y^2 = 0$
29. $2x^2 + 9xy + 14y^2 - 5 = 0$
30. $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
31. $2x^2 + 4\sqrt{3}xy + 6y^2 + \sqrt{3}x - y = 0$

32. Identify the equation $2x^2 + 4xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y + 12 = 0$ as a circle, ellipse, parabola, or hyperbola. Then find θ to the nearest degree.

The graph of each equation is a degenerate case. Identify the graph and then draw it.

33. $(x - 2)^2 - (x + 3)^2 = 5(y + 2)$

34. $2x^2 + 6y^2 + 8x - 12y + 14 = 0$

35. $y^2 - 9x^2 = 0$

36. $(x - 2)^2 + (y - 2)^2 + 4(x + y) = 8$

Graphing Calculator

Use the Quadratic Formula to solve each equation for *y*. Then use a graphing calculator to draw the graph.

37. $x^2 - 2xy + y^2 - 5x - 5y = 0$

38. $2x^2 + 9xy + 14y^2 = 5$

39. $8x^2 + 5xy - 4y^2 = -2$

40. $2x^2 + 4\sqrt{3}xy + 6y^2 + 3x = y$

41. $2x^2 + 4xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y = -12$

42. $9x^2 + 4xy + 6y^2 = 20$

Applications and Problem Solving

43. Agriculture Many farmers in the Texas panhandle divide their land into smaller one square mile units. This square is then divided into four smaller units, or quadrants, of equal size. Each quadrant contains its own central pivot irrigation system. Suppose the center of the square mile of land is the origin.

a. Determine the translation in feet needed to place an irrigation system in Quadrant I.

b. Determine the equation for the path of the outer end of the irrigation system in Quadrant I.

44. Critical Thinking Identify the graph of each equation. Then determine the minimum angle of rotation needed to transform each equation so that the rotated graph coincides with its original graph.

a. $x^2 + 6x + 9 - y = 0$

b. $8x^2 + 6y^2 = 24$

c. $4xy = 25$

d. $15x^2 + 15y^2 = 60$

45. Critical Thinking Prove that a circle with equation $x^2 + y^2 = r^2$ remains unchanged under any rotation θ.

46. Astronomy Suppose the equation $31x^2 - 10\sqrt{3}xy + 21y^2 = 144$ models the shape of a reflecting mirror in a telescope.

a. Determine whether the reflector in the telescope is elliptical, parabolic, or hyperbolic.

b. Using a graphing calculator, sketch the graph of the equation.

c. Determine the angle through which the mirror has been rotated.

47. Critical Thinking Consider the equation $9x^2 - 2\sqrt{3}xy + 11y^2 - 24 = 0$.

a. Determine the minimum angle of rotation needed to transform the graph of this equation to a graph whose axes are on the x- and y-axes.

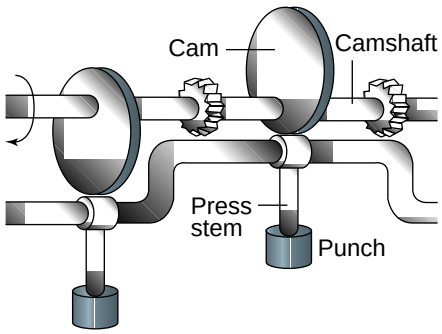
b. Use the angle of rotation in part **a** to find the new equation of the graph.

48. Manufacturing A cam in a punch press is shaped like an ellipse with the equation $\frac{x^2}{81} + \frac{y^2}{36} = 1$. The camshaft goes through the focus on the positive axis.

a. Graph a model of the cam.

b. Find an equation that translates the model so that the camshaft is at the origin.

c. Find the equation of the model in part b when the cam is rotated to an upright position.



Mixed Review

49. Identify the conic section represented by the equation $5y^2 - 3x^2 + 4x - 3y - 100 = 0$. *(Lesson 10-6)*

50. Write the equation of the ellipse that has its center at $(2, -3)$, $a = 1$, and $e = \frac{2\sqrt{6}}{5}$. *(Lesson 10-3)*

51. Graph $r = \frac{1}{\cos(\theta + 15°)}$. *(Lesson 9-4)*

52. Boating A boat heads due west across a lake at 8 m/s. If a current of 5 m/s moves due south, what is the boat's resultant velocity? *(Lesson 8-1)*

53. Which value is greater, cos 70° or cos 170°? *(Lesson 6-3)*

54. Change $\frac{5\pi}{16}$ radians to degree measure to the nearest minute. *(Lesson 5-1)*

55. Decompose $\frac{2y + 5}{y^2 + 3y + 2}$ into partial fractions. *(Lesson 4-6)*

56. If y varies inversely as x and $y = 4$ when $x = 12$, find y when $x = 5$. *(Lesson 3-8)*

57. Solve the system of equations algebraically. *(Lesson 2-2)*

$$8m - 3n - 4p = 6$$
$$4m + 9n - 2p = -4$$
$$6m + 12n + 5p = -1$$

58. Graph $h(x) = [\![x]\!] - 3$. *(Lesson 1-7)*

59. SAT Practice If $1 < b < 2$ and $2 < a < 3$, which statement is true about the expression $\frac{5a^8b^5}{180a^6b^2}$?

A The value of the expression is never greater than 1.

B The value of the expression is always between $\frac{1}{9}$ and 2.

C The value of the expression is always greater than 1.

D The value of the expression is always between $\frac{1}{36}$ and $\frac{3}{2}$.

E The value of the expression is always between 0 and $\frac{1}{2}$.

Extra Practice See p. A45.

10-8

Systems of Second-Degree Equations and Inequalities

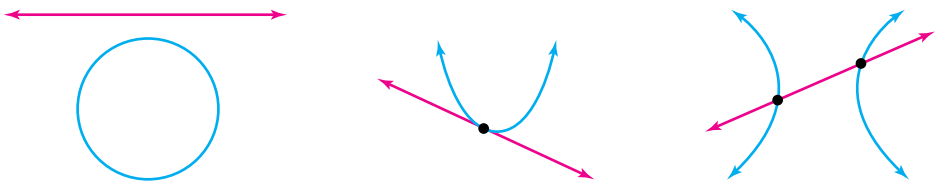
OBJECTIVE
- Graph and solve systems of second degree equations and inequalities.

Real World Application

SEISMOLOGY The principal use of a seismograph network is to locate the epicenters of earthquakes. Seismograph stations in Chihuahua, Mazatlan, and Rosarito, Mexico, form one such network. Suppose this network detects an earthquake 622 kilometers from the Chihuahua station, 417 kilometers from the Mazatlan station, and 582 kilometers from the Rosarito station. Seismographic networks use complex software to approximate the location of the epicenter. The intersection of the three circles on the map shows the location of the epicenter to be near La Paz, Mexico. *You will solve a problem related to this in Exercise 40.*

The equation of each circle in the application above is a second-degree equation. So the three circles represent a system of second-degree equations. The coordinates of the point that satisfies all three equations is the solution to the system, which is the location of the earthquake's epicenter. You can solve this system graphically by locating the point where all three circles intersect.

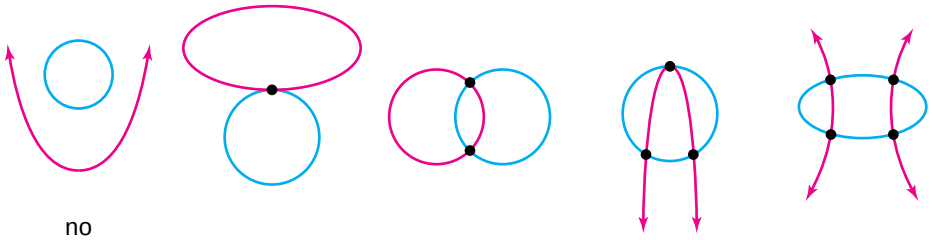
The number of solutions of a system of second-degree equations equals the number of times the graphs of the equations intersect. If the system is composed of a line and a conic, there may be 0, 1, or 2 solutions. *Remember that a line is a degenerate conic.*



no solution | 1 solution | 2 solutions

CONTENTS

If the system is composed of two conics, there may be 0, 1, 2, 3, or 4, solutions.



While you can determine the number of solutions by graphing the equations of a system, the exact solution is not always apparent. To find the exact solution, you must use algebra.

Example 1

a. Graph the system of equations. Use the graph to find approximate solutions.

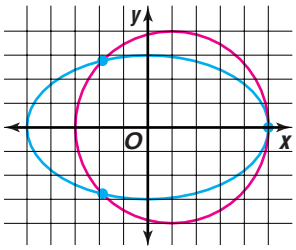
b. Solve the system algebraically.

$$9x^2 + 25y^2 = 225$$
$$x^2 + y^2 - 2x = 15$$

Graphing Calculator Tip

You can use the CALC feature on a graphing calculator to approximate the intersection of any two functions graphed.

a. The graph of the first equation is an ellipse. The graph of the second equation is a circle. When a system is composed of an ellipse and a circle, there may be 0, 1, 2, 3, or 4 possible solutions. Graph each equation. There appear to be 3 solutions close to (5, 0), (−2, −3), and (−2, 3).



b. Since both equations contain a single term involving y, y^2, you can solve the system as follows.

First, multiply each side of the second equation by −25.

$$-25(x^2 + y^2 - 2x) = -25(15)$$
$$-25x^2 - 25y^2 + 50x = -375$$

Then, add the equations.

$$\begin{array}{rr} 9x^2 + 25y^2 & = 225 \\ -25x^2 - 25y^2 + 50x & = -375 \\ \hline -16x^2 + 50x & = -150 \end{array}$$

Using the quadratic formula, $x = 5$ or $-\frac{15}{8}$.

Now find y by substituting these values for x in one of the original equations.

$$x^2 + y^2 - 2x = 15$$
$$5^2 + y^2 - 2(5) = 15 \quad x = 5$$
$$25 + y^2 - 10 = 15$$
$$y^2 = 0$$
$$y = 0$$

$$x^2 + y^2 - 2x = 15$$
$$\left(-\frac{15}{8}\right)^2 + y^2 - 2\left(-\frac{15}{8}\right) = 15 \quad x = -\frac{15}{8}$$
$$\frac{225}{64} + y^2 + \frac{15}{4} = 15$$
$$y^2 = \frac{495}{64}$$
$$y = \pm\frac{3\sqrt{55}}{8}$$

The solutions are (5, 0), $\left(-\frac{15}{8}, \frac{3\sqrt{55}}{8}\right)$, and $\left(-\frac{15}{8}, -\frac{3\sqrt{55}}{8}\right)$. *How do the decimal approximations of these values compare to the approximations made in part* ***a****?*

Systems of second-degree equations are useful in solving real-world problems involving more than one parameter.

Example 2 **SALES** **During the month of January, Photo World collected $2700 from the sale of a certain camera. After lowering the price by $15, the store sold 30 more cameras and took in $3375 from the sale of this camera the next month.**

a. Write a system of second-degree equations to model this situation.

b. Find the price of the camera during each month.

c. Use a graphing calculator to check your solution.

a. From the information in the problem, we can write two equations, each of which is the equation of a conic section.

Let x = number of cameras sold
Let y = price per camera in January.

Sales in January: $xy = 2700$
Sales in February: $(x + 30)(y - 15) = 3375$

These are equations of hyperbolas.

b. To solve the system algebraically, use substitution. You can rewrite the equation of the first hyperbola as $y = \frac{2700}{x}$. Before substituting, expand the left-hand side of the second equation and simplify the equation.

$$(x + 30)(y - 15) - 3375 = 0$$
$$xy - 15x + 30y - 450 - 3375 = 0 \quad \textit{Expand } (x - 30)(y + 15).$$
$$xy - 15x + 30y - 3825 = 0 \quad \textit{Simplify.}$$
$$2700 - 15x + 30\left(\frac{2700}{x}\right) - 3825 = 0 \quad xy = 2700,\ y = \frac{2700}{x}$$
$$-15x - 1125 + \frac{81{,}000}{x} = 0 \quad \textit{Simplify.}$$
$$-15x^2 - 1125x + 81{,}000 = 0 \quad \textit{Multiply each side by } x.$$
$$x^2 + 75x - 5400 = 0 \quad \textit{Divide each side by } -15.$$
$$(x - 45)(x + 120) = 0 \quad \textit{Factor.}$$
$$x - 45 = 0 \quad \text{or} \quad x + 120 = 0$$
$$x = 45 \qquad\qquad x = -120$$

Since the number of cameras sold cannot be negative, the store sold 45 cameras during January.

The price of each camera sold during January was $\$\frac{2700}{45}$ or \$60, and the price per camera in February was \$60 − 15 or \$45.

CONTENTS

c. Solve each equation for y. Then, graph the equations on the same screen.

$$y = \frac{2700}{x}$$

$$y = \frac{3375}{x + 30} + 15$$

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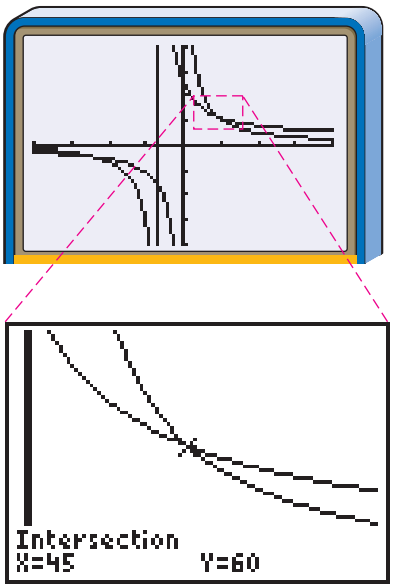


Use ZOOM to enlarge the section of the graph containing the intersection in the first quadrant. Use the **CALC: intersect** function to find the coordinates of the solution, (45, 60).

Look Back
You can refer to Lesson 2-6 to review solving systems of linear inequalities.

Previously you learned how to graph different types of inequalities by graphing the corresponding equation and then testing points in the regions of the graph to find solutions for the inequality. The same process is used when graphing systems of inequalities involving second-degree equations.

Example 3 **Graph the solutions for the system of inequalities.**

$$x^2 + 4y^2 \leq 4$$
$$x^2 > y^2 + 1$$

First graph $x^2 + 4y^2 \leq 4$. The ellipse should be a solid curve. Test a point either inside or outside the ellipse to see if its coordinates satisfy the inequality.

Test (0, 0):

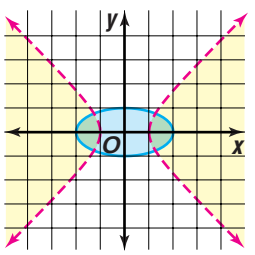
$$x^2 + y^2 \leq 4$$
$$0^2 + 4(0)^2 \stackrel{?}{\leq} 4 \quad (x, y) = (0, 0)$$
$$0 \leq 4 \quad \checkmark$$

Since (0, 0) satisfies the inequality, shade the interior of the ellipse. Then graph $x^2 \geq y^2 + 1$. The hyperbola should be dashed. Test a point inside the branches of the hyperbola or outside its branches. *Since a hyperbola is symmetric, you need not test points within both branches.*

Test (2, 0):

$$x^2 > y^2 + 1$$
$$2^2 \stackrel{?}{>} 0^2 + 1 \quad (x, y) = (2, 0)$$
$$4 > 1 \quad \checkmark$$

Since (2, 0) satisfies the inequality, the regions inside the branches should be shaded. The intersection of the two graphs, which is shown in green, represents the solution of the system.



Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Draw** figures illustrating each of the possible numbers of solutions to a system involving the equations of a parabola and a hyperbola.
2. **Write** a system of equations involving two different conic sections that has exactly one solution, the origin.
3. **Describe** the graph of a system of second-degree equations having infinitely many solutions.
4. *Math Journal* **Write** a paragraph explaining how to solve a system of second-degree inequalities.

Guided Practice

Solve each system of equations algebraically. Round to the nearest tenth. Check the solutions by graphing each system.

5. $\dfrac{(x-1)^2}{20} + \dfrac{(y-1)^2}{5} = 1$
   $x - y = 0$

6. $x^2 + y^2 = 16$
   $x + 2y = 10$

7. $9x^2 - 4y^2 = 36$
   $x^2 + y^2 = 4$

8. $x^2 = y$
   $xy = 1$

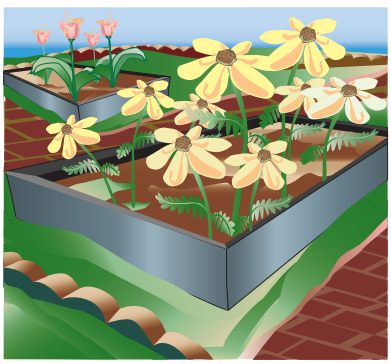
Graph each system of inequalities.

9. $x^2 + y^2 \geq 16$
   $x + y \leq 2$

10. $(x-5)^2 + 2y < 10$
    $y - 9 \geq -2x$

11. $x^2 + y^2 \leq 100$
    $x^2 + y^2 \geq 25$

12. **Gardening** A garden contains two square flowerbeds. The total area of the flowerbeds is 680 square feet, and the second bed has 288 more square feet than the first.
    a. Write a system of second-degree equations that models this situation.
    b. Graph the system found in part **a** and estimate the solution.
    c. Solve the system algebraically to find the length of each flowerbed within the garden.



Exercises

Practice

Solve each system of equations algebraically. Round to the nearest tenth. Check the solutions by graphing each system.

13. $x - 1 = 0$
    $y^2 = 49 - x^2$

14. $xy = 2$
    $x^2 = 3 + y^2$

15. $4x^2 + y^2 = 25$
    $-1 = 2x + y$

16. $x - y = 2$
    $x^2 = 100 - y^2$

17. $x - y = 0$
    $\dfrac{(x-1)^2}{9} - y^2 = 1$

18. $3x^2 = 9 - y^2$
    $x^2 + 2y^2 = 10$

19. $(y-1)^2 = 4 + x$
    $x + y = -1$

20. $x^2 + y^2 = 13$
    $xy + 6 = 0$

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CONTENTS

21. $x^2 + 4y^2 = 36$
$x^2 + y - 3 = 0$

22. $x^2 = 16 - y^2$
$2y - x + 3 = 0$

23. Find the coordinates of the point(s) of intersection for the graphs of $x^2 = 25 - 9y^2$ and $xy = -4$.

Graph each system of inequalities.

24. $x + y^2 \leq 9$
$y + x^2 \leq 0$

25. $x^2 + 4y^2 < 16$
$x^2 \leq y^2 + 4$

26. $x^2 + y^2 \leq 36$
$x + y^2 > 0$

27. $y^2 < 81 - 9x^2$
$16 \leq x^2 + y^2$

28. $y + 4 < (x - 3)^2$
$y^2 + x \geq 5$

29. $16x^2 + 49y^2 \leq 784$
$49x^2 + 16y^2 \geq 784$

30. $x - (y - 1)^2 \leq 0$
$4y^2 \geq x^2 - 16$

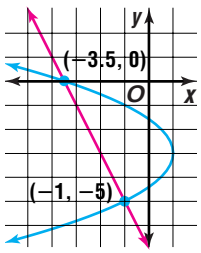
31. $y - x^2 < 2$
$4x^2 + 9y^2 > 36$

32. $x > \frac{2}{y}$
$16x^2 - 25y^2 \geq 400$

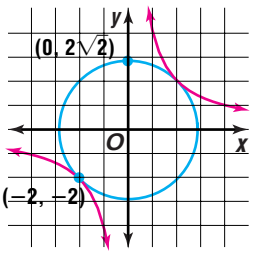
33. Graph the solution to the system $(x + 3)^2 + (y + 2)^2 \geq 36$ and $x + 3 = 0$.

Write the system of equations or inequalities represented by each graph.

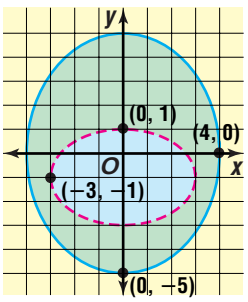
34.



35.



36.



Applications and Problem Solving

37. Construction Carrie has 150 meters of fencing material to make a pen for her bird dog. She wants to form a rectangular pen with an area of 800 square meters. What will be the dimensions of her pen?

a. Let x be the width of the field and y be its length. Write a system of equations that models this situation.

b. How many solutions are possible for this type of system?

c. Graph the system to estimate the dimensions of the pen.

d. Solve this system algebraically, rounding the dimensions to the nearest tenth of a meter.

38. Engineering The Transport and Road Research Laboratory in Great Britain proposes the use of parabolic speed bumps 4 inches in height and 1 foot in width.

a. Write a system of second-degree inequalities that models a cross-section of this speed bump. Locate the vertex of the speed bump at (0, 4).

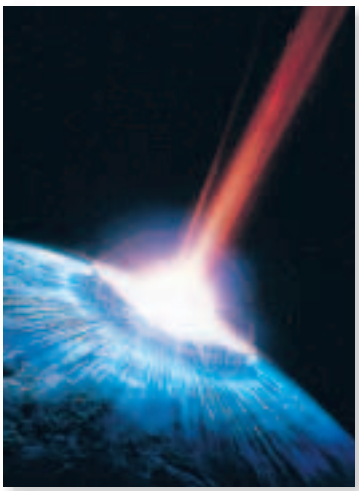
b. Graph the system found in part **a**.

c. If the height of the speed bump is decreased to 3 inches, write a system of equations to model this new cross-section.

39. Critical Thinking Solve the system $x = -y + 1$, $xy = -12$, and $y^2 = 25 - x^2$ algebraically. Then graph the system to verify your solution(s).

40. **Seismology** Each of three stations in a seismograph network has detected an earthquake in their region. Seismograph readings indicate that the epicenter of the earthquake is 50 kilometers from the first station, 40 kilometers from the second station, and 13 kilometers from the third station. On a map in which each grid represents one square kilometer, the first station is located at the origin, the second station at (0, 30), and the third station at (35, 18).
   a. Write a system of second-degree equations that models this situation.
   b. Graph the system and use the graph to approximate the location of the epicenter.
   c. Solve the system of equations algebraically to determine the location of the epicenter.

41. **Critical Thinking** Find the value of k so that the graphs of $x = 2y^2$ and $x + 3y = k$ are tangent to each other.



42. **Entertainment** In a science fiction movie, astronomers track a large incoming asteroid and predict that it will strike Earth with disastrous results. Suppose a certain latitude of Earth's surface is modeled by $x^2 + y^2 = 40$ and the path of the asteroid is modeled by $x = 0.25y^2 + 5$.
   a. Graph the two equations on the same axes.
   b. Will the asteroid strike Earth? If so, what are the coordinates of the point of impact?
   c. Describe this situation with parametric equations. Assume both the asteroid and Earth's surface are moving counterclockwise.
   d. Graph the equations found in part **c** using a graphing calculator. Use a window that shows complete graphs of both Earth's surface and the asteroid's path.

Mixed Review

43. Find the equation of $\frac{x^2}{9} + y^2 = 1$ after a 30° rotation about the origin. *(Lesson 10-7)*

44. Write an equation in standard form of the line with the parametric equations $x = 4t + 1$ and $y = 5t - 7$. *(Lesson 8-6)*

45. Simplify $4 \csc \theta \cos \theta \tan \theta$. *(Lesson 7-1)*

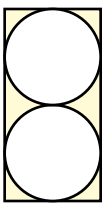
46. **Mechanics** A pulley of radius 10 centimeters turns at 5 revolutions per second. Find the linear velocity of the belt driving the pulley in meters per second. *(Lesson 6-2)*

47. Determine between which consecutive integers the real zeros of the function $f(x) = x^3 - 4$ are located. *(Lesson 4-5)*

48. Graph $y = (x + 2)^2 - 3$ and its inverse. *(Lesson 3-4)*

49. Is the relation $\{(4, 0), (3, 0), (5, -2), (4, -3), (0, -13)\}$ a function? Explain. *(Lesson 1-1)*

50. **SAT/ACT Practice** In the figure at the right, two circles are tangent to each other and each is tangent to three sides of the rectangle. If the radius of each circle is 2, what is the area of the shaded region?



   A $32 - 12\pi$ **B** $16 - 8\pi$ **C** $16 - 6\pi$
   D $8 - 6\pi$ **E** $32 - 8\pi$

Extra Practice See p. A45.

CONTENTS

10-8B Shading Areas on a Graph

An Extension of Lesson 10-8

OBJECTIVE
- Graph a system of second-degree inequalities using the **Shade(** command.

The **Shade(** command can be used to shade areas between the graphs of two equations. To shade an area on a graph, select **7:Shade(** from the **DRAW** menu. The instruction is pasted to the home screen. The argument, or restrictive information, for this command is as follows.

Shade(*lowerfunc,upperfunc,Xleft,Xright,pattern,patres*)

This command draws the lower function, *lowerfunc,* and the upper function, *upperfunc,* in terms of *X* on the current graph and shades the area that is specifically above *lowerfunc* and below *upperfunc.* This means that only the areas between the two functions defined are shaded.

Xleft and *Xright,* if included, specify left and right boundaries for the shading. *Xleft* and *Xright* must be numbers between **Xmin** and **Xmax**, which are the defaults.

The parameter *pattern* specifies one of four shading patterns.

pattern = **1** vertical lines (default)
pattern = **2** horizontal lines
pattern = **3** 45° lines with positive slope
pattern = **4** 45° lines with negative slope

The parameter *patres* specifies one of eight shading resolutions.

patres = **1** shades every pixel (default)
patres = **2** shades every second pixel
patres = **3** shades every third pixel
patres = **4** shades every fourth pixel
patres = **5** shades every fifth pixel
patres = **6** shades every sixth pixel
patres = **7** shades every seventh pixel
patres = **8** shades every eighth pixel

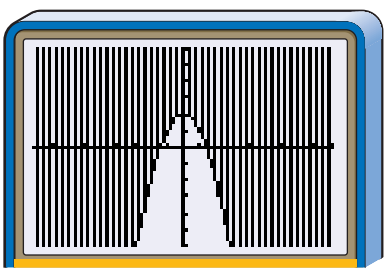
In a system of second-degree inequalities, this technique can be used to shade the interior region of a conic that is not a function.

Example

Graph the solutions for the system of inequalities below.

$y \geq -x^2 + 2$
$x^2 + 9y^2 \leq 36$

The boundary equation of the first inequality, $y = -x^2 + 2$, is a function. This inequality is graphed by first entering the equation $y = -x^2 + 2$ into the **Y=** list. Since the test point (0, 3) satisfies the inequality, set the graph style to ◥ (shade above).

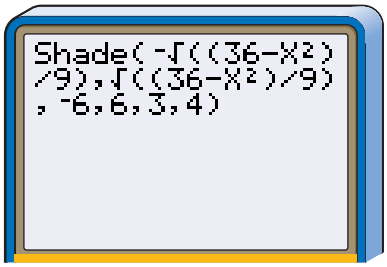


[−9.1, 9.1] scl:1 by [−6, 6] scl:1

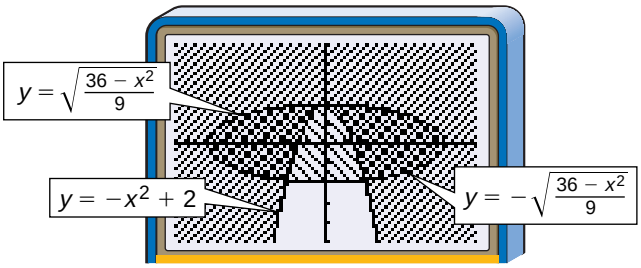
(continued on the next page)

The boundary equation for the second inequality is $x^2 + 9y^2 = 36$, which is defined using two functions, $y = -\sqrt{\frac{36 - x^2}{9}}$ and $y = \sqrt{\frac{36 - x^2}{9}}$. The lower function is $y = -\sqrt{\frac{36 - x^2}{9}}$, and the upper function is $y = \sqrt{\frac{36 - x^2}{9}}$. The two halves of the ellipse intersect at $x = -6$ and $x = 6$.

The expression to shade the area between the two halves of the ellipse is shown at the right.



Pressing ENTER will compute the graph as shown below.



[−9.1, 9.1] scl:1 by [−6, 6] scl:1

The solution set for this system of inequalities is the darker region in which the shadings for the two inequalities over lap.

To clear the any **Shade(** commands from the viewing window, select **1:ClrDraw** from the **DRAW** menu and then press ENTER. Remember to also clear any functions defined in the **Y=** list.

TRY THESE

Use the shade feature to graph each system of second-degree inequalities.

1. $y \le x^2 - 5$
$9y^2 - x^2 \le 36$

2. $x^2 + y^2 \le 16$
$x \ge y^2 - 4$

3. $16x^2 + 25y^2 \le 400$
$25x^2 - 16y^2 \ge 400$

4. $8x^2 + 32y^2 \le 256$
$32x^2 + 8y^2 \le 256$

WHAT DO YOU THINK?

5. Recall that the **Shade(** command can only shade the area between two functions.

a. To shade just the solution set for the system of inequalities in the example problem, how many regions would need a separate **Shade(** command?

b. How could you determine the approximate domain intervals for each region?

c. List and then execute three **Shade(** commands to shade the region representing the solution set for the example problem.

6. Use the **Shade(** command to create a "real-world" picture. Make a list of each command needed to create the picture, as well as a sketch of what the finished picture should look like.

VOCABULARY

analytic geometry (p. 618)
asymptotes (p. 642)
axis of symmetry (p. 653)
center (p. 623, 642)
circle (p. 623)
concentric (p. 623)
conic section (p. 623)
conjugate axis (p. 642)
degenerate conic (p. 623)
directrix (p. 653)
eccentricity (p. 636)
ellipse (p. 631)
equilateral hyperbola (p. 647)
focus (p. 631, 642, 653)
hyperbola (p. 642)
locus (p. 658)
major axis (p. 631)
minor axis (p. 631)
radius (p. 623)
rectangular hyperbola (p. 648)
semi-major axis (p. 632)
semi-minor axis (p. 632)
transverse axis (p. 642)
vertex (p. 631, 642, 653)

UNDERSTANDING AND USING THE VOCABULARY

State whether each statement is *true* or *false.* If false, replace the underlined word(s) to make a true statement.

1. Circles, ellipses, parabolas, and hyperbolas are all examples of conic sections.
2. Circles that have the same radius are concentric circles.
3. The line segment connecting the vertices of a hyperbola is called the conjugate axis.
4. The foci of an ellipse are located along the major axis of the ellipse.
5. In the general form of a circle, A and C have opposite signs.
6. A parabola is symmetric with respect to its vertex.
7. The shape of an ellipse is described by a measure called eccentricity.
8. A hyperbola is the set of all points in a plane that are the same distance from a given point and a given line.
9. The general equation of a rectangular hyperbola, where the coordinate axes are the asymptotes, is $xy = c$.
10. A point is the degenerate form of a parabola conic.

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 10-1 Find the distance and midpoint between two points on a coordinate plane.

- Find the distance between points at (3, 8) and (−5, 10).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-5 - 3)^2 + (10 - 8)^2}$$
$$= \sqrt{68} \text{ or } 2\sqrt{17}$$

REVIEW EXERCISES

Find the distance between each pair of points with the given coordinates. Then, find the midpoint of the segment that has endpoints at the given coordinates.

11. (1, −6), (−3, −4)

12. (a, b), $(a + 3, b + 4)$

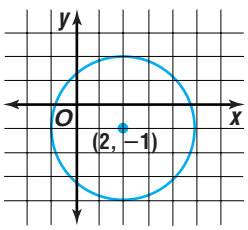
13. Determine whether the points $A(-5, -2)$, $B(3, 4)$, $C(10, 3)$, and $D(2, -3)$ are the vertices of a parallelogram. Justify your answer.

Lesson 10-2 Determine the standard form of the equation of a circle and graph it.

- Write $x^2 + y^2 - 4x + 2y - 4 = 0$ in standard form. Then graph the equation.

$$x^2 + y^2 - 4x + 2y - 4 = 0$$
$$x^2 - 4x + ? + y^2 + 2y + ? = 4$$
$$x^2 - 4x + 4 + y^2 + 2y + 1 = 4 + 4 + 1$$
$$(x - 2)^2 + (y + 1)^2 = 9$$

The center of the circle is located at (2, −1), and the radius is 3.



Write the standard form of the equation of each circle described. Then graph the equation.

14. center at (0, 0), radius $3\sqrt{3}$

15. center at (2, 1), tangent to the y-axis

Write the standard form of each equation. Then graph the equation.

16. $x^2 + y^2 = 6y$

17. $x^2 + 14x + y^2 + 6y = 23$

18. $3x^2 + 3y^2 + 6x + 12y - 60 = 0$

19. Write the standard form of the equation of the circle that passes through points at (1, 1), (−2, 2), and (−5, 1). Then identify the center and radius.

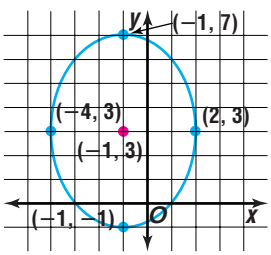
Lesson 10-3 Determine the standard form of the equation of an ellipse and graph it.

- For the equation, $\frac{(x + 1)^2}{9} + \frac{(y - 3)^2}{16} = 1$, find the coordinates of the center, foci, and vertices of the ellipse. Then graph the equation.

center: (−1, 3)

foci: $\left(-1, 3 + \sqrt{7}\right)$, $\left(-1, 3 - \sqrt{7}\right)$

vertices: (2, 3), (−4, 3), (−1, −1), (−1, 7)



For the equation of each ellipse, find the coordinates of the center, foci, and vertices. Then graph the equation.

20. $\frac{(x - 5)^2}{16} + \frac{(y - 2)^2}{36} = 1$

21. $4x^2 + 25y^2 - 24x + 50y = 39$

22. $6x^2 + 4y^2 + 24x - 32y + 64 = 0$

23. $x^2 + 4y^2 + 124 = 8x + 48y$

24. Write the equation of an ellipse centered at (−4, 1) with a vertical semi-major axis 9 units long and a semi-minor axis 6 units long.

OBJECTIVES AND EXAMPLES

Lesson 10-4 Determine the standard and general forms of the equation of a hyperbola and graph it.

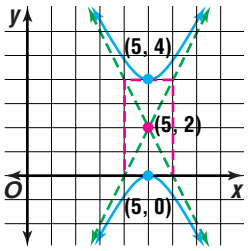
Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $\frac{(y-2)^2}{4} - (x-5)^2 = 1$. Then graph the equation.

center: (5, 2)

foci: $\left(5, 2+\sqrt{5}\right)$, $\left(5, 2-\sqrt{5}\right)$

vertices: (5, 4), (5, 0)

asymptotes: $y - 2 = \pm 2(x-5)$



REVIEW EXERCISES

For the equation of each hyperbola, find the coordinates of the center, the foci, and the vertices and the equations of the asymptotes of its graph. Then graph the equation.

25. $\frac{x^2}{25} - \frac{y^2}{16} = 1$

26. $\frac{(y+5)^2}{36} - \frac{(x-1)^2}{9} = 1$

27. $x^2 - 4y^2 - 16y = 20$

28. $9x^2 - 16y^2 - 36x - 96y + 36 = 0$

29. Graph $xy = 9$.

Write an equation of the hyperbola that meets each set of conditions.

30. The length of the conjugate axis is 10 units, and the vertices are at (1, −1) and (1, 5).

31. The vertices are at (−2, −3) and (6, −3), and a focus is at (−4, −3).

Lesson 10-5 Determine the standard and general forms of the equation of a parabola and graph it.

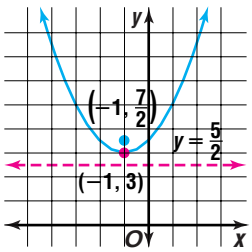
For the equation $(x+1)^2 = 2(y-3)$, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

vertex: (−1, 3)

focus: $\left(-1, \frac{7}{2}\right)$

directrix: $y = \frac{5}{2}$

axis of symmetry: $x = -1$



For the equation of each parabola, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

32. $(x-5)^2 = 8(y-3)$

33. $(y+2)^2 = -16(x-1)$

34. $y^2 + 6y - 4x = -25$

35. $x^2 + 4x = y - 8$

Write an equation of the parabola that meets each set of conditions.

36. The parabola passes through the point at (−3, 7), has its vertex at (−1, 3), and opens to the left.

37. The focus is at (5, 2), and the equation of the directrix is $y = -4$.

Lesson 10-6 Recognize conic sections in their rectangular form by their equations.

Identify the conic section represented by the equation $2x^2 - 3x - y + 4 = 0$.

$A = 2$ and $C = 0$

Since $C = 0$, the conic is a parabola.

Identify the conic section represented by each equation.

38. $5x^2 - 7x + 2y^2 = 10$

39. $xy = 5$

40. $2x^2 + 4x + 2y^2 - 6y + 16 = 0$

41. $4y^2 + 6x - 5y = 20$

OBJECTIVES AND EXAMPLES

Lesson 10-6 Find a rectangular equation for a curve defined parametrically and vice versa.

Find the rectangular equation of the curve whose parametric equations are $x = 3 \sin t$ and $y = \cos t$, where $0 \le t \le 2\pi$. Then graph the equation using arrows to indicate orientation.

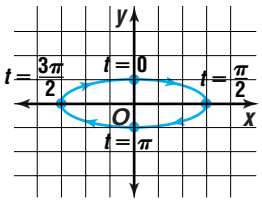
$x = 3\sin t \Rightarrow \sin t = \frac{x}{3}$

$y = \cos t \Rightarrow \cos t = y$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{3}\right)^2 + y^2 = 1$

$\frac{x^2}{9} + y^2 = 1$



REVIEW EXERCISES

Find the rectangular equation of the curve whose parametric equations are given. Then graph the equation using arrows to indicate orientation.

42. $x = t, y = -t^2 + 3, -\infty \le t \le \infty$

43. $x = \cos 4t, y = \sin 4t, 0 \le t \le \frac{\pi}{2}$

44. $x = 2 \sin t, y = 3 \cos t, 0 \le t \le 2\pi$

45. $x = \sqrt{t}, y = \frac{t}{2} - 1, 0 \le t \le 9$

Find parametric equations for each rectangular equation.

46. $y = 2x^2 + 4$

47. $x^2 + y^2 = 49$

48. $\frac{x^2}{36} + \frac{y^2}{81} = 1$

49. $x = -y^2$

Lesson 10-7 Find the equations of conic sections that have been translated or rotated and find the angle of rotation for a given equation.

To find the equation of a conic section with respect to a rotation of θ, replace

x with $x' \cos \theta + y' \sin \theta$
and y with $-x' \sin \theta + y' \cos \theta$.

Identify the graph of each equation. Write an equation of the translated or rotated graph in general form.

50. $4x^2 + 9y^2 = 36, \theta = \frac{\pi}{6}$

51. $y^2 - 4x = 0, \theta = 45°$

52. $4x^2 - 16(y - 1)^2 = 64$ for $T_{(1, -2)}$

Identify the graph of each equation. Then find θ to the nearest degree.

53. $6x^2 + 2\sqrt{3}xy + 8y^2 = 45$

54. $x^2 - 6xy + 9y^2 = 7$

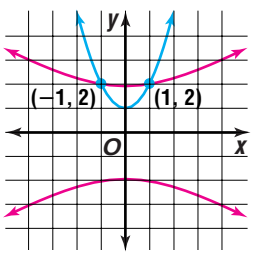
Lesson 10-8 Graph and solve systems of second degree equations and inequalities.

Solve the system of equations $x^2 - y = -1$ and $x^2 - 3y^2 = -11$ algebraically.

$$\begin{array}{r} x^2 - y + 1 = 0 \\ -x^2 + 3y^2 - 11 = 0 \\ \hline 3y^2 - y - 10 = 0 \end{array}$$

$y = -\frac{5}{3}$ or $y = 2$

Substituting we find the solutions to be (1, 2) and (−1, 2). The graph shows these solutions to be true.



Solve each system of equations algebraically. Round to the nearest tenth. Check the solutions by graphing each system.

55. $(x - 1)^2 + 4(y - 1)^2 = 20$
$x = y$

56. $2x - y = 0$
$y^2 = 49 + x^2$

57. $x^2 - 4x - 4y = 4$
$(x - 2)^2 + 4y = 0$

58. $x^2 + y^2 = 12$
$xy = -4$

Graph each system of inequalities.

59. $x^2 + y \le 4$
$y^2 - x \le 0$

60. $xy \ge 9$
$x^2 + y^2 < 36$

61. $x^2 \le 16 - y^2$
$36y^2 > 324 - 9x^2$

62. $x^2 - 4y \ge 8$
$4y^2 - 25x^2 \ge 100$

CONTENTS

APPLICATIONS AND PROBLEM SOLVING

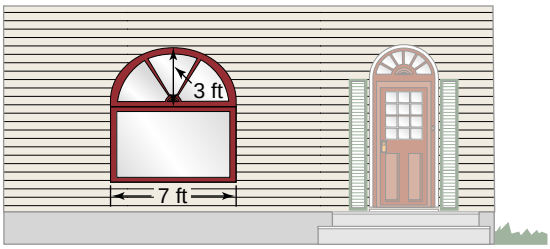
63. Gardening Migina bought a new sprinkler that covers part or all of a circular area. With the center of the sprinkler as the origin, the sprinkler sends out water far enough to reach a point located at (12, 16). *(Lesson 10-2)*

a. Find an equation representing the farthest points the sprinkler can reach.

b. Migina's backyard is 40 feet wide and 50 feet long. If Migina waters her backyard without moving the sprinkler, what percent of her backyard will not be watered directly?

64. Astronomy A satellite orbiting Earth follows an elliptical path with Earth at its center. The eccentricity of the orbit is 0.2, and the major axis is 12,000 miles long. Assuming that the center of the ellipse is the origin and the foci lie on the x-axis, write the equation of the orbit of the satellite. *(Lesson 10-6)*

65. Carpentry For a remodeling project, a carpenter is building a picture window that is topped with an arch in the shape of a semi-ellipse. The width of the window is to be 7 feet, and the height of the arch is to be 3 feet. To sketch the arch above the window, the carpenter uses a 7-foot string attached to two thumbtacks. Approximately where should the thumbtacks be placed? *(Lesson 10-3)*



ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. An ellipse has its center at the origin and an eccentricity of $\frac{1}{9}$. What is a possible equation for the ellipse?
2. A parabola has an axis of symmetry of $x = 2$ and a focus of (2, 5). What is a possible equation for the parabola in standard form?

PORTFOLIO

Choose one of the conic sections you studied in this chapter. Explain why it is a conic section and describe how you graph it.

Additional Assessment See page A65 for Chapter 10 practice test.

Unit 3 internet Project

SPACE—THE FINAL FRONTIER

Out in Orbit!

- Search the internet for a satellite, space vehicle, or planet that travels in an orbit around a planet or star.
- Find data on the orbit of the satellite, space vehicle, or planet. This information should include the closest and farthest distance of that object from the planet it is orbiting.
- Make a scale drawing of the orbit of the satellite, space vehicle, or planet. Label important features and dimensions.
- Write a summary describing the orbit of the satellite, space vehicle, or planet. Be sure to discuss which conic section best models the orbit.

Ratio and Proportion Problems

THE PRINCETON REVIEW

A ratio compares a part to a part. A fraction compares a part to a whole.

Ratios only tell you the *relative* sizes of quantities, not the actual quantities.

When setting up a proportion, label quantities to prevent careless errors.

Several problems on the SAT and ACT involve ratios or proportions. The ratio of x to y can be expressed in several ways.

$$\frac{x}{y} \qquad x{:}y \qquad x \text{ to } y$$

Think of a ratio as comparing parts of a whole. If the ratio of boys to girls in a class is 2:1, then one part is 2, one part is 1, and the whole is 3. The fraction of boys in the class is $\frac{2}{3}$.

Memorize the property of proportions.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

ACT EXAMPLE

1. The ratio of boys to girls in a class is 4 to 5. If there are a total of 27 students in the class, how many boys are in the class?

A 4 **B** 9 **C** 12 **D** 14 **E** 17

HINT Notice what question is asked. Is it a ratio, a fraction, or a number?

Solution The ratio is 4 to 5, so the whole must be 9. The fraction of boys is $\frac{4}{9}$. The total number of students is 27, so the number of boys is $\frac{4}{9}$ of 27 or 12. The answer is choice **C**.

Alternate Solution Another method is to use a 'ratio box' to record the numbers and guide your calculations.

Boys	Girls	Whole
4	5	9
		27

In the Whole column, 9 must be multiplied by 3 to get the total of 27. So multiply the 4 by 3 to get the number of boys.

Boys	Girls	Whole
4	5	9
12		27

This is answer choice **C**.

SAT EXAMPLE

2. If 2 packages contain a total of 12 doughnuts, how many doughnuts are there in 5 packages?

A 12 **B** 24 **C** 30 **D** 36 **E** 60

HINT In a proportion, one ratio equals another ratio.

Solution Write a proportion.

packages → $\frac{2}{12} = \frac{5}{x}$ ← packages

doughnuts → ← doughnuts

Cross multiply.

$$\frac{2}{12} = \frac{5}{x}$$
$$2(x) = 5(12)$$
$$x = 30$$

The answer is choice **C**.

Alternate Solution You can also solve this problem without using a proportion. Since 2 packages contain 12 doughnuts, each package must contain $12 \div 2$ or 6 doughnuts. Then five packages will contain 5×6 or 30 doughnuts.

This is answer choice **C**.

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

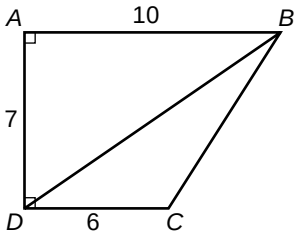
1. In a jar of red and green jelly beans, the ratio of green jelly beans to red jelly beans is 5:3. If the jar contains a total of 160 jelly beans, how many of them are red?

   A 30 **B** 53 **C** 60 **D** 100 **E** 160

2. If $a^2b = 12^2$ and b is an odd integer, then a could be divisible by all of the following EXCEPT

   A 3 **B** 4 **C** 6 **D** 9 **E** 12

3. In the figure below, $\angle A$ and $\angle ADC$ are right angles, the length of $\overline{AD}$ is 7 units, the length of $\overline{AB}$ is 10 units, and the length of $\overline{DC}$ is 6 units. What is the area, in square units, of $\triangle DCB$?

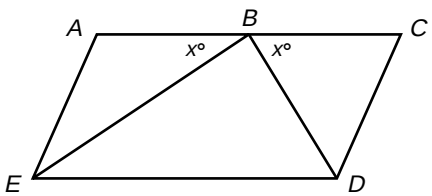
   

   A 21
   B 24
   C $3\sqrt{149}$
   D 142
   E 210

4. A science class has a ratio of girls to boys of 4 to 3. If the class has a total of 35 students, how many more girls are there than boys?

   A 20 **B** 15 **C** 7 **D** 5 **E** 1

5. In the figure below, $\overline{AC} \parallel \overline{ED}$. If the length of $BD = 3$, what is the length of BE?

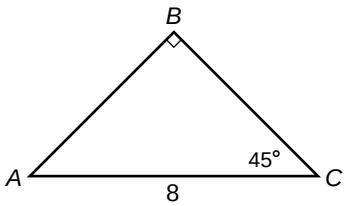
   Note: Figure not drawn to scale.

   A 3 **B** 4 **C** 5 **D** $3\sqrt{3}$
   E It cannot be determined from the information given.

6. What is the slope of the line that contains points at (6, 4) and (13, 5)?

   A $\frac{1}{8}$ **B** $-\frac{1}{9}$ **C** $\frac{1}{7}$ **D** 1 **E** 7

7. In $\triangle ABC$ below, if AC is equal to 8, then BC is equal to

   A $8\sqrt{2}$ **B** 8 **C** 6
   D $4\sqrt{2}$ **E** $3\sqrt{2}$

8. The ratio of $\frac{1}{7}$ to $\frac{1}{5}$ is equal to the ratio of 100 to

   A $\frac{20}{7}$ **B** 20 **C** 35 **D** 100 **E** 140

9. If there are 4 more nickels in a jar than there are dimes, which could be the ratio of dimes to nickels in the jar?

   A $\frac{8}{10}$
   B 1
   C $\frac{14}{10}$
   D 4
   E None of the above

10. **Grid-In** Twenty bottles contain a total of 8 liters of apple juice. If each bottle contains the same amount of apple juice, how much juice (in liters) is in each bottle?

internet CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**