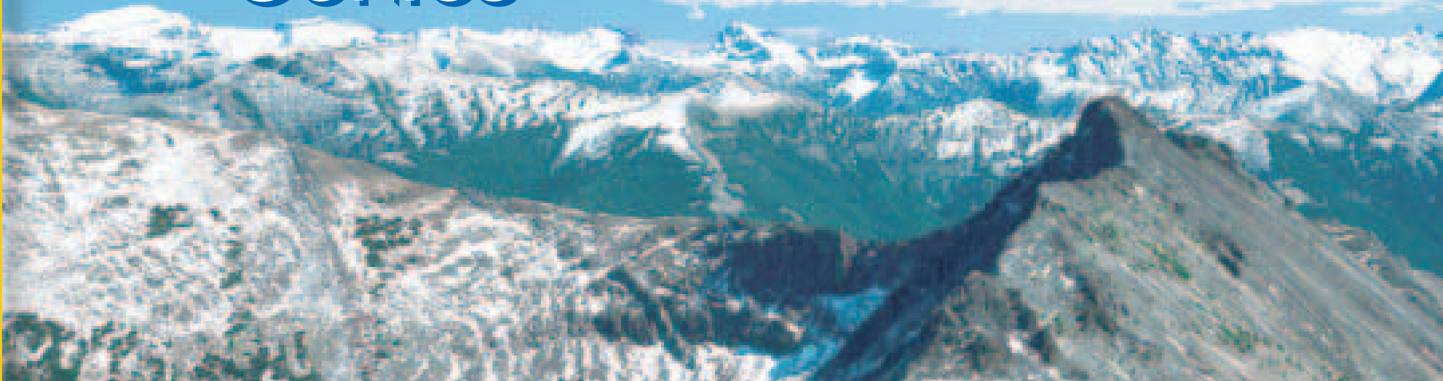
Chapter 10

CONICS



CHAPTER OBJECTIVES

- **Use analytic methods to prove geometric relationships.** *(Lesson 10-1)*
- **Use the standard and general forms of the equations of circles, parabolas, ellipses, and hyperbolas.** *(Lessons 10-2, 10-3, 10-4, and 10-5)*
- **Graph circles, parabolas, ellipses, and hyperbolas.** *(Lessons 10-2, 10-3, 10-4, and 10-5)*
- **Find the eccentricity of conic sections.** *(Lessons 10-2, 10-3, 10-4, and 10-5)*
- **Recognize conic sections by their equations.** *(Lesson 10-6)*
- **Find parametric equations for conic sections defined by rectangular equations and vice versa.** *(Lesson 10-6)*
- **Find the equations of conic sections that have been translated or rotated.** *(Lesson 10-7)*
- **Graph and solve systems of second-degree equations and inequalities.** *(Lesson 10-8)*

10-1

Introduction to Analytic Geometry

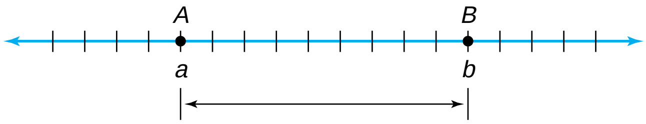
OBJECTIVES

- Find the distance and midpoint between two points on a coordinate plane.
- Prove geometric relationships among points and lines using analytical methods.

Real World Application

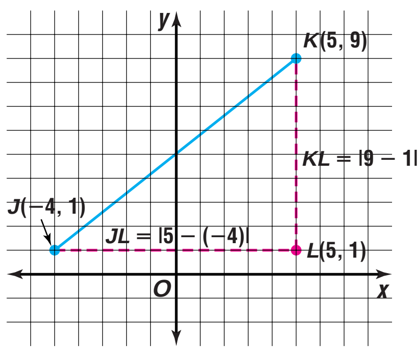
SEARCH AND RESCUE The *Absaroka Search Dogs* has provided search teams, each of which consists of a canine and its handler, to assist in lost person searches throughout the Montana and Wyoming area since 1986. While the dogs use their highly sensitive noses to detect the lost individual, the handlers use land navigation skills to insure that they do not become lost themselves. A handler needs to be able to read a map and calculate distances. *A problem related to this will be solved in Example 2.*

The distance between two points on a number line can be found by using absolute value. Let A and B be two points with coordinates a and b, respectively.



Distance between A and B $\Rightarrow$ $|a - b|$ or $|b - a|$

The distance between two points in the coordinate plane can also be found. Consider points $J(-4, 1)$ and $K(5, 9)$. To find JK, first choose a point L such that $\overline{JL}$ is parallel to the x-axis and $\overline{KL}$ is parallel to the y-axis. In this case, L has coordinates (5, 1). Since K and L lie along the line $x = 5$, KL is equal to the absolute value of the difference in the y-coordinates of K and L, $|9 - 1|$. Similarly, JL is equal to the absolute value of the difference in the x-coordinates of J and L, $|5 - (-4)|$.



Since $\triangle JKL$ is a right triangle, JK can be found using the Pythagorean Theorem.

$$(JK)^2 = (KL)^2 + (JL)^2$$ *Pythagorean Theorem*

$$JK = \sqrt{(KL)^2 + (JL)^2}$$ *Take the positive square root of each side.*

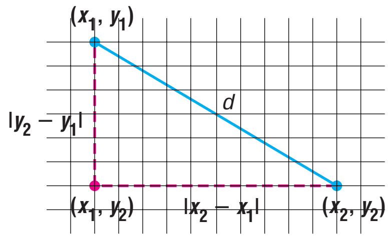
$$JK = \sqrt{|9 - 1|^2 + |5 - (-4)|^2}$$ *$KL = |9 - 1|$, $JL = |15 - (-4)|$*

$$JK = \sqrt{8^2 + 9^2}$$ *$|9 - 1|^2 = 8^2$, $|15 - (-4)|^2 = 9^2$*

$$JK = \sqrt{145} \text{ or about } 12$$

$\overline{JK}$ is about 12 units long.

From this specific case, we can derive a formula for the distance between any two points. In the figure, assume (x_1, y_1) and (x_2, y_2) represent the coordinates of any two points in the plane.



$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$ *Pythagorean Theorem*

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Why does $|x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$?*

Distance Formula for Two Points

The distance, d units, between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Examples

1 Find the distance between points at (−3, 7) and (2, −5).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Distance Formula*

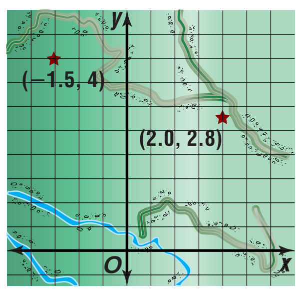
$d = \sqrt{(2 - (-3))^2 + (-5 - 7)^2}$ *Let $(x_1, y_1) = (-3, 7)$ and $(x_2, y_2) = (2, -5)$.*

$d = \sqrt{5^2 + (-12)^2}$

$d = \sqrt{169}$ or 13

The distance is 13 units.

2 SEARCH AND RESCUE Refer to the application at the beginning of the lesson. Suppose a backpacker lies injured in the region shown on the map at the right. Each side of a square on the grid represents 15 meters. An Absaroka team searching for the missing individual is located at (−1.5, 4.0) on the map grid while the injured person is located at (2.0, 2.8). How far is the search team from the missing person?



Use the distance formula to find the distance between (2.0, 2.8) and (−1.5, 4.0).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

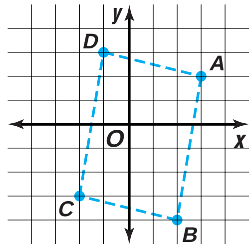
$d = \sqrt{(-1.5 - 2.0)^2 + (4.0 - 2.8)^2}$ *Let $(x_1, y_1) = (2.0, 2.8)$ and $(x_2, y_2) = (-1.5, 4.0)$.*

$d = 13.69$ or 3.7

The map distance is 3.7 units. Each unit equals 15 kilometers. So, the actual distance is about 3.7(15) or 55.5 kilometers.

You can use the Distance Formula and what you know about slope to investigate geometric figures on the coordinate plane.

Example 3 **Determine whether quadrilateral *ABCD* with vertices *A*(3, 2), *B*(2, −4), *C*(−2, −3), and *D*(−1, 3) is a parallelogram.**



Recall that a quadrilateral is a parallelogram if one pair of opposite sides are parallel and congruent.

First, graph the figure. $\overline{DA}$ and $\overline{CB}$ are one pair of opposite sides.

To determine if $\overline{DA} \parallel \overline{CB}$, find the slopes of $\overline{DA}$ and $\overline{CB}$.

slope of $\overline{DA}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \textit{Slope formula}$$

$$= \frac{2 - 3}{3 - (-1)} \qquad \textit{Let } (x_1, y_1) = (-1, -3) \textit{ and } (x_2, y_2) = (3, 2).$$

$$= -\frac{1}{4}$$

slope of $\overline{CB}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \textit{Slope formula}$$

$$= \frac{-4 - (-3)}{2 - (-2)} \qquad \textit{Let } (x_1, y_1) = (-2, -3) \textit{ and } (x_2, y_2) = (2, -4).$$

$$= -\frac{1}{4}$$

Their slopes are equal. Therefore, $\overline{DA} \parallel \overline{CB}$.

To determine if $\overline{DA} \cong \overline{CB}$, use the distance formula to find *DA* and *CB*.

$$DA = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad CB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{[3 - (-1)]^2 + (2 - 3)^2} \qquad = \sqrt{[2 - (-2)]^2 + [-4 - (-3)]^2}$$

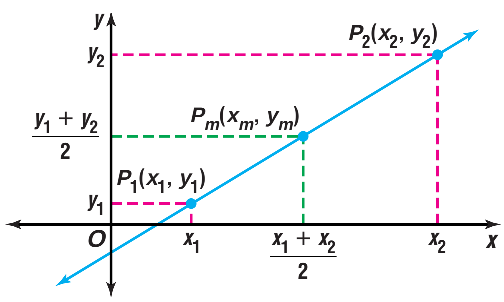
$$= \sqrt{17} \qquad = \sqrt{17}$$

The measures of $\overline{DA}$ and $\overline{CB}$ are equal. Therefore, $\overline{DA} \cong \overline{CB}$.

Since $\overline{DA} \parallel \overline{CB}$ and $\overline{DA} \cong \overline{CB}$, quadrilateral *ABCD* is a parallelogram. *You can also check your work by showing* $\overline{DC} \parallel \overline{AB}$ *and* $\overline{DC} \cong \overline{AB}$*.*

Look Back
Refer to Lesson 1-3 to review the slope formula.

In addition to finding the distance between two points, you can use the coordinates of two points to find the midpoint of the segment between the points. In the figure at the right, the midpoint of $\overline{P_1P_2}$ is P_m. Notice that the x-coordinate of P_m is the average of the x-coordinates of P_1 and P_2. The y-coordinate of P_m is the average of the y-coordinates of P_1 and P_2.



Midpoint of a Line Segment

If the coordinates of P_1 and P_2 are (x_1, y_1) and (x_2, y_2), respectively, then the midpoint of $\overline{P_1P_2}$ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 4 **Find the coordinates of the midpoint of the segment that has endpoints at (−2, 4) and (6, −5).**

Let (−2, 4) be (x_1, y_1) and (6, −5) be (x_2, y_2). Use the Midpoint Formula.

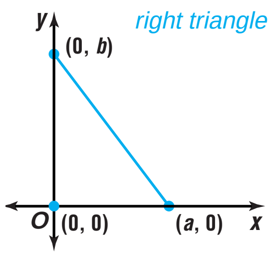
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-2 + 6}{2}, \frac{4 + (-5)}{2}\right)$$
$$= \left(2, -\frac{1}{2}\right)$$

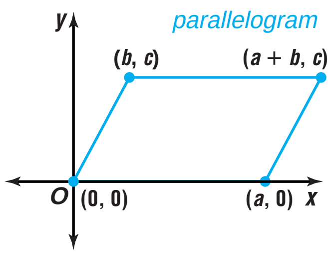
The midpoint of the segment is at $\left(2, -\frac{1}{2}\right)$.

Many theorems from plane geometry can be more easily proven by analytic methods. That is, they can be proven by placing the figure in a coordinate plane and using algebra to express and draw conclusions about the geometric relationships. The study of coordinate geometry from an algebraic perspective is called **analytic geometry.**

When using analytic methods to prove theorems from geometry, the position of the figure in the coordinate plane can be arbitrarily selected as long as size and shape are preserved. This means that the figure may be translated, rotated, or reflected from its original position. For polygons, one vertex is usually located at the origin, and one side coincides with the *x*-axis, as shown below.

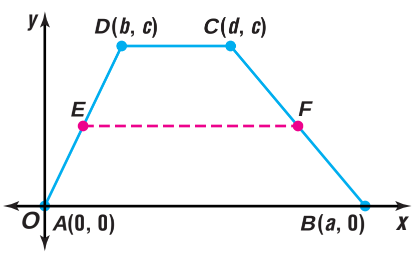
In a right triangle, the legs are on the axes.





Example 5 **Prove that the measure of the median of a trapezoid is equal to one half of the sum of the measures of the two bases.**

In trapezoid *ABCD*, choose two vertices as $A(0, 0)$ and $B(a, 0)$. Since $\overline{AB} \parallel \overline{DC}$, $\overline{DC}$ lies on a horizontal grid line of the coordinate plane. Therefore, *C* and *D* must have the same *y*-coordinate. Choose arbitrary letters to represent the *y*-coordinates, and the two *x*-coordinates; in this case, $D(b, c)$ and $C(d, c)$. Let *E* be the midpoint of $\overline{AD}$, and let *F* be the midpoint of $\overline{BC}$.



Now, find the coordinates of E and F by using the Midpoint Formula.

The coordinates of E are $\left(\frac{b+0}{2}, \frac{c+0}{2}\right)$ or $\left(\frac{b}{2}, \frac{c}{2}\right)$.

The coordinates of F are $\left(\frac{d+a}{2}, \frac{c+0}{2}\right)$ or $\left(\frac{d+a}{2}, \frac{c}{2}\right)$.

Then find the measures of each base and the median by using the Distance Formula.

$DC = \sqrt{(d-b)^2 + (c-c)^2}$ or $d - b$

$AB = \sqrt{(a-0)^2 + (0-0)^2}$ or a

$EF = \sqrt{\left(\frac{d+a}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2}$ or $\frac{1}{2}(d - b + a)$

Calculate one half of the sum of the measures of the bases.

$\frac{1}{2}(DC + AB) = \frac{1}{2}(d - b + a)$ $\quad DC = d - b,\ AB = a$

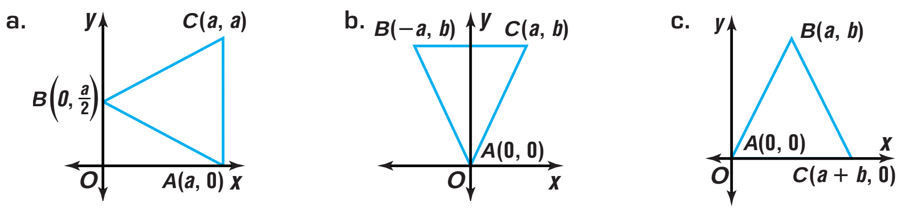
Since both $\frac{1}{2}(DC + AB)$ and EF equal $\frac{1}{2}(d - b + a)$, it follows that $\frac{1}{2}(DC + AB) = EF$. Therefore, the measure of the median of a trapezoid is one half of the sum of the measures of its bases.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why only the positive square root is considered when applying the distance formula.
2. **Describe** how can you show that a midpoint of a segment is equidistant from its endpoints given the coordinates of each point.
3. **Determine** whether each diagram represents an isosceles triangle. Explain your reasoning.



4. **Describe** four different ways of proving that a quadrilateral is a parallelogram if you are given the coordinates of its vertices.

Guided Practice

Find the distance between each pair of points with the given coordinates. Then, find the coordinates of the midpoint of the segment that has endpoints at the given coordinates.

5. (5, 1), (5, 11) 6. (0, 0), (−4, −3) 7. (−2, 2), (0, 4)

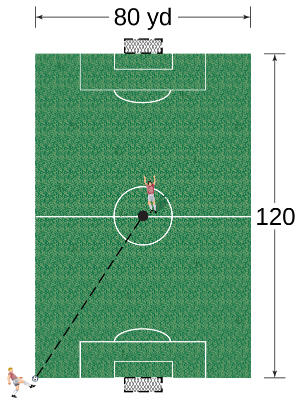
8. Determine whether the quadrilateral *ABCD* with vertices $A(3, 4)$, $B(6, 2)$, $C(8, 7)$, and $D(5, 9)$ is a parallelogram. Justify your answer.

9. Determine whether the triangle *XYZ* with vertices $X(-3, 2)$, $Y(-1, -6)$, and $Z(5, 0)$ is isosceles. Justify your answer.

10. Consider rectangle *ABCD*.
   a. Draw and label rectangle *ABCD* on the coordinate plane.
   b. Prove that $\overline{AC} \cong \overline{BD}$.
   c. Suppose the diagonals intersect at point *E*. Prove that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$.
   d. What can you conclude about the diagonals of a rectangle? Explain.

Crew Stadium, Columbus, Ohio

11. **Sports** The dimensions of a soccer field are120 yards by 80 yards. A player kicks the ball from a corner to his teammate at the center of the playing field. Suppose the kicker is located at the origin.
   a. Find the ordered pair that represents the location of the kicker's teammate.
   b. Find the distance the ball travels.



EXERCISES

Practice

Find the distance between each pair of points with the given coordinates. Then, find the coordinates of the midpoint of the segment that has endpoints at the given coordinates.

12. $(-1, 1), (4, 13)$
13. $(1, 3), (-1, -3)$
14. $(8, 0), (0, 8)$
15. $(-1, -6), (5, -3)$
16. $\left(3\sqrt{2}, -5\right), \left(7\sqrt{2}, -1\right)$
17. $(a, 7), (a, -9)$
18. $(6 + r, s), (r - 2, s)$
19. $(c, d), (c + 2, d - 1)$
20. $(w - 2, w), (w, 4w)$

21. Find all values of *a* so that the distance between points at $(a, -9)$ and $(-2a, 7)$ is 20 units.

22. If $M\left(-3, \frac{5}{2}\right)$ is the midpoint of $\overline{CD}$ and *C* has coordinates $(4, -1)$, find the coordinates of *D*.

Determine whether the quadrilateral having vertices with the given coordinates is a parallelogram.

23. $(-2, 3), (-3, -2), (2, -3), (3, 2)$
24. $(4, 11), (8, 14), (4, 19), (0, 15)$

25. Collinear points lie on the same line. Find the value of *k* for which the points $(15, 1)$, $(-3, -8)$, and $(3, k)$ are collinear.

26. Determine whether the points $A(-3, 0)$, $B\left(-1, 2\sqrt{3}\right)$, and $C(1, 0)$ are the vertices of an equilateral triangle. Justify your answer.

27. Show that points $E(2, 5)$, $F(4, 4)$, $G(2, 0)$, and $H(0, 1)$ are the vertices of a rectangle.

interNET CONNECTION

Graphing Calculator Programs
To download a graphing calculator program that determines the distance and midpoint between two points, visit **www.amc.glencoe.com**

Prove using analytic methods. Be sure to include a coordinate diagram.

28. The measure of the line segment joining the midpoints of two sides of a triangle is equal to one-half the measure of the third side.

29. The diagonals of an isosceles trapezoid are congruent.

30. The medians to the congruent sides of an isosceles triangle are congruent. (*Hint*: A median of a triangle is a segment connecting a vertex to the midpoint of the side opposite the vertex.)

31. The diagonals of a parallelogram bisect each other.

32. The line segments joining the midpoints of consecutive sides of any quadrilateral form a parallelogram.

Applications and Problem Solving

33. Geometry The vertices of a rectangle are at $(-3, 1)$, $(-1, 3)$, $(3, -1)$, and $(1, -3)$. Find the area of the rectangle.

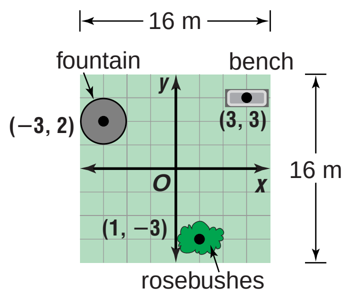
34. Web Page Design Many Internet Web pages are designed so that when the cursor is positioned over a specified area of an image, lines of text are displayed. The programming string "$x \mid y \mid$ width $\mid$ height" defines the location of the bottom left corner of the designated region using x and y coordinates and then defines the width and height of the region in pixels.

a. If the center of a Web page is located at the origin, graph the two regions defined by $-22 \mid 12 \mid 10 \mid 8$ and $31 \mid -10 \mid 8 \mid 10$.

b. Suppose the two regions are to be no less than 40 pixels apart. Calculate the distance between the regions at their closest points to determine if this criteria is met.

35. Critical Thinking Prove analytically that the segments joining midpoints of consecutive sides of an isosceles trapezoid form a rhombus. Include a coordinate diagram with your proof.

36. Landscaping The diagram shows the plans made by a landscape artist for a homeowner's 16-meter by 16-meter backyard. The homeowner has requested that the rosebushes, fountain, and garden bench be placed so that they are no more than 14 meters apart.



a. Has the landscape artist met the homeowner's requirements? Explain.

b. The homeowner has purchased a sundial to be placed midway between the fountain and the rosebushes. Determine the coordinates indicating where the sundial should be placed.

37. Critical Thinking Consider point $M(t, 3t - 12)$.

a. Prove that for all values of t, M is equidistant from $A(0, 3)$ and $B(9, 0)$.

b. Describe the figure formed by the points M for all values of t. What is the relationship between the figure and points A and B?

Mixed Review

38. Find $(-5 + 12\boldsymbol{i})^2$. *(Lesson 9-8)*

Extra Practice See p. A44.

CONTENTS

39. **Physics** Suppose that during a storm, the force of the wind blowing against a skyscraper can be expressed by the vector $\langle 115, 2018, 0 \rangle$, where each measure in the ordered triple represents the force in Newtons. What is the magnitude of this force? *(Lesson 8-3)*

40. Verify that $2\sec^2 x = \frac{1}{1+\sin x} + \frac{1}{1-\sin x}$ is an identity. *(Lesson 7-2)*

41. A circle has a radius of 12 inches. Find the degree measure of the central angle subtended by an arc 11.5 inches long. *(Lesson 6-1)*

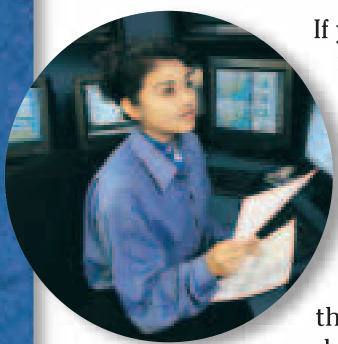
42. Find $\sin 390°$. *(Lesson 5-3)*

43. Solve $z^2 - 8z = -14$ by completing the square. *(Lesson 4-2)*

44. **SAT Practice** **Grid-In** If $x^2 = 16$ and $y^2 = 4$, what is the greatest possible value of $(x - y)^2$?

CAREER CHOICES

Meteorologist



If you find the weather intriguing, then you may want to investigate a career in meteorology. Meteorologists spend their time studying weather and forecasting changes in the weather. They analyze charts, weather maps, and other data to make predictions about future weather patterns. Meteorologists also research different types of weather, such as tornadoes and hurricanes, and may even teach at universities.

As a meteorologist, you may choose to specialize in one of several areas such as climatololgy, operational meteorology, or industrial meteorology. As a meteorologist, you might even be seen on television forecasting the weather for your area!

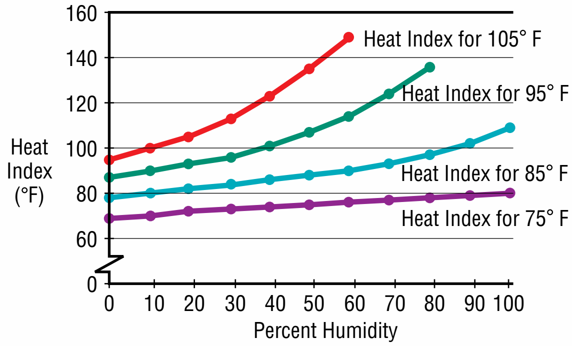
Career Overview

Degree Preferred:
Bachelor's degree in meteorology

Related Courses:
mathematics, geography, physics, computer science

Outlook:
slower than average job growth through the year 2006

Heat Index for Various Temperatures and Humidities



Source: *The World Almanac* 1999

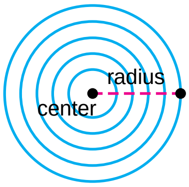
interNET CONNECTION For more information on careers in meteorology, visit: **www.amc.glencoe.com**

10-2 Circles

OBJECTIVES
- Use and determine the standard and general forms of the equation of a circle.
- Graph circles.

Real World Application

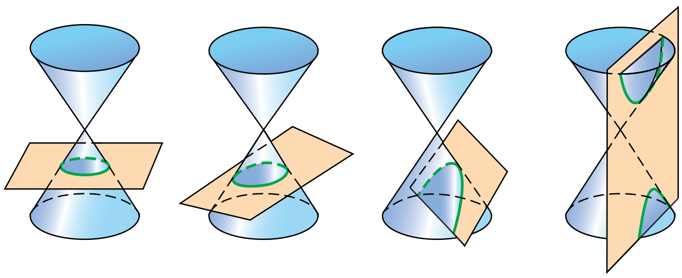
SEISMOLOGY Portable autonomous digital seismographs (PADSs) are used to investigate the strong ground motions produced by the aftershocks of large earthquakes. Suppose a PADS is deployed 2 miles west and 3.5 miles south of downtown Olympia, Washington, to record the aftershocks of a recent earthquake. While there, the PADS detects and records the seismic activity of another quake located 24 miles away. What are all the possible locations of this earthquake's epicenter? *This problem will be solved in Example 2.*



The pattern of the shock waves from an earthquake form **concentric** circles. A **circle** is the set of all points in the plane that are equidistant from a given point in the plane, called the **center.** The distance from the center to any point on the circle is called the **radius** of the circle. Concentric circles have the same center but not necessarily the same radius.

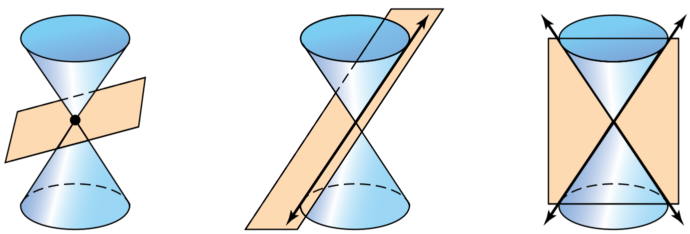
A circle is one type of **conic section.** Conic sections, which include circles, parabolas, ellipses and hyperbolas, were first studied in ancient Greece sometime between 600 and 300 B.C. The Greeks were largely concerned with the properties, not the applications, of conics. In the seventeenth century, applications of conics became prominent in the development of calculus.

Conic sections are used to describe all of the possible ways a plane and a double right cone can intersect. In forming the four basic conics, the plane does not pass through the vertex of the cone.



circle ellipse parabola hyperbola

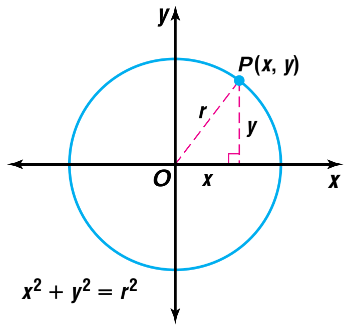
When the plane does pass through the vertex of a conical surface, as illustrated below, the resulting figure is called a **degenerate conic.** A degenerate conic may be a point, line, or two intersecting lines.



point (degenerate ellipse) line (degenerate parabola) intersecting lines (degenerate hyperbola)

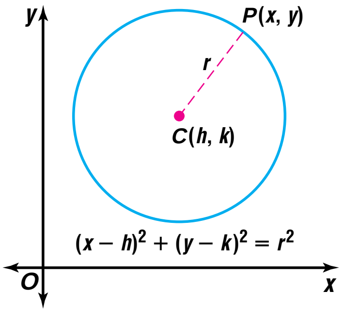
Radius can also refer to the line segment from the center to any point on the circle.

In the figure at the right, the center of the circle is at the origin. By drawing a perpendicular from any point $P(x, y)$ on the circle but not on an axis to the x-axis, you form a right triangle. The Pythagorean Theorem can be used to write an equation that describes every point on a circle whose center is located at the origin.



$x^2 + y^2 = r^2$ *Pythagorean Theorem*

This is the equation for the parent graph of all circles.



Suppose the center of this circle is translated from the origin to $C(h, k)$. You can use the distance formula to write the equation for this translated circle.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Distance formula*

$r = \sqrt{(x - h)^2 + (y - k)^2}$ *$d = r$, $(x_2, y_2) = (x, y)$, and $(x_1, y_1) = (h, k)$*

$r^2 = (x - h)^2 + (y - k)^2$ *Square each side.*

This equation is the standard form of the equation of a circle.

Standard Form of the Equation of a Circle	The standard form of the equation of a circle with radius r and center at (h, k) is $$(x - h)^2 + (y - k)^2 = r^2.$$

Examples

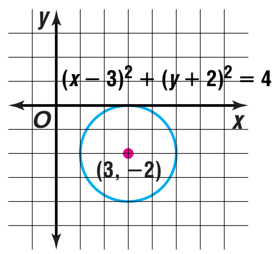
1 **Write the standard form of the equation of the circle that is tangent to the x-axis and has its center at $(3, -2)$. Then graph the equation.**

Since the circle is tangent to the x-axis, the distance from the center to the x-axis is the radius. The center is 2 units below the x-axis. Therefore, the radius is 2.

$(x - h)^2 + (y - k)^2 = r^2$ *Standard form*

$(x - 3)^2 + [y - (-2)]^2 = 2^2$ *$h = 3$, $k = -2$, $r = 2$*

$(x - 3)^2 + (y + 2)^2 = 4$



The standard form of the equation for this circle is $(x - 3)^2 + (y + 2)^2 = 4$.

2 **SEISMOLOGY** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Write an equation for the set of points representing all possible locations of the earthquake's epicenter. Let downtown Olympia, Washington, be located at the origin.

b. Graph the equation found in part a.

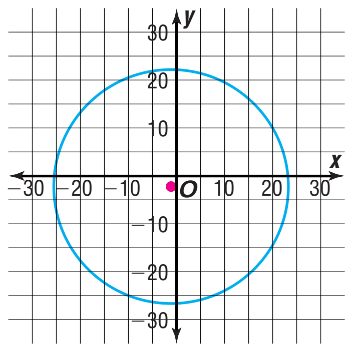
Graphing Calculator Tip

To graph a circle on a graphing calculator, first solve for y. Then graph the two resulting equations on the same screen by inserting $\{1, -1\}$ in front of the equation for the symbol $\pm$. Use **5:ZSquare** in the **ZOOM** menu to make the graph look like a circle

a. The location of the PADS, 2 miles west and 3.5 miles south of downtown Olympia, Washington, can be expressed as the ordered pair $(-2, -3.5)$. Since any point 24 miles from the seismograph could be the epicenter, the radius of the circle is 24.

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Standard form}$$
$$[x - (-2)]^2 + [y - (-3.5)]^2 = 24^2 \quad (h, k) = (-2, -3.5) \text{ and } r = 24$$
$$(x + 2)^2 + (y + 3.5)^2 = 576$$

b. The location of the epicenter lies on the circle with equation $(x + 2)^2 + (y + 3.5)^2 = 576$.



$(x + 2)^2 + (y + 3.5)^2 = 576$

The standard form of the equation of a circle can be expanded to obtain a general form of the equation.

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Standard form}$$
$$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = r^2 \quad \text{Expand } (x - h)^2 \text{ and } (y - k)^2.$$
$$x^2 + y^2 + (-2h)x + (-2k)y + (h^2 + k^2) - r^2 = 0$$

Since h, k, and r are constants, let D, E, and F equal $-2h$, $-2k$ and $(h^2 + k^2) - r^2$, respectively.

$$x^2 + y^2 + Dx + Ey + F = 0$$

This equation is called the general form of the equation of a circle.

General Form of the Equation of a Circle

The general form of the equation of a circle is

$$x^2 + y^2 + Dx + Ey + F = 0,$$

where D, E, and F are constants.

Notice that the coefficients of x^2 and y^2 in the general form must be 1. If those coefficients are not 1, division can be used to transform the equation so that they are 1. Also notice that there is no term containing the product of the variables, xy.

When the equation of a circle is given in general form, it can be rewritten in standard form by completing the square for the terms in x and the terms in y.

Example 3 **The equation of a circle is $2x^2 + 2y^2 - 4x + 12y - 18 = 0$.**

a. Write the standard form of the equation.

b. Find the radius and the coordinates of the center.

c. Graph the equation.

a. $2x^2 + 2y^2 - 4x + 12y - 18 = 0$

$x^2 + y^2 - 2x + 6y - 9 = 0$ *Divide each side by 2.*

$(x^2 - 2x + ?) + (y^2 + 6y + ?) = 9$ *Group to form perfect square trinomials.*

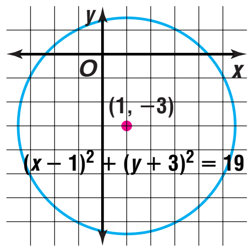
$(x^2 - 2x + 1) + (y^2 + 6y + 9) = 9 + 1 + 9$ *Complete the square.*

$(x - 1)^2 + (y + 3)^2 = 19$ *Factor the trinomials.*

$(x - 1)^2 + (y + 3)^2 = \left(\sqrt{19}\right)^2$ *Express 19 as $\left(\sqrt{19}\right)^2$ to show that $r = \sqrt{19}$.*

b. The center of the circle is located at (1, −3), and the radius is $\sqrt{19}$.

c. Plot the center at (−1, 3). The radius of $\sqrt{19}$ is approximately equal to 4.4.



From geometry, you know that any two points in the coordinate plane determine a unique line. It is also true that any three noncollinear points in the coordinate plane determine a unique circle. The equation of this circle can be found by substituting the coordinates of the three points into the general form of the equation of a circle and solving the resulting system of three equations.

Example 4 **Write the standard form of the equation of the circle that passes through the points at (5, 3), (−2, 2), and (−1, −5). Then identify the center and radius of the circle.**

Substitute each ordered pair for (x, y) in $x^2 + y^2 + Dx + Ey + F = 0$, to create a system of equations.

$(5)^2 + (3)^2 + D(5) + E(3) + F = 0$ *$(x, y) = (5, 3)$*

$(-2)^2 + (2)^2 + D(-2) + E(2) + F = 0$ *$(x, y) = (-2, 2)$*

$(-1)^2 + (-5)^2 + D(-1) + E(-5) + F = 0$ *$(x, y) = (-1, -5)$*

Simplify the system of equations.

$$5D + 3E + F + 34 = 0$$
$$-2D + 2E + F + 8 = 0$$
$$-D - 5E + F + 26 = 0$$

The solution to the system is $D = -4$, $E = 2$, and $F = -20$.

The general form of the equation of the circle is $x^2 + y^2 - 4x + 2y - 20 = 0$. After completing the square, the standard form is $(x - 2)^2 + (y + 1)^2 = 25$. The center of the circle is at (2, −1), and its radius is 5.

Look Back
Refer to Lesson 2-2 to review solving systems of three equations.

CONTENTS

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how to convert the general form of the equation of a circle to the standard form of the equation of a circle.
2. **Write** the equations of five concentric circles with different radii whose centers are at $(-4, 9)$.
3. **Describe** how you might determine the equation of a circle if you are given the endpoints of the circle's diameter.
4. **Find a counterexample** to this statement: The graph of any equation of the form $x^2 + y^2 + Dx + Ey + F = 0$ is a circle.
5. **You Decide** Kiyo says that you can take the square root of each side of an equation. Therefore, he decides that $(x - 3)^2 + (y - 1)^2 = 49$ and $(x - 3) + (y - 1) = 7$ are equivalent equations. Ramon says that the equations are not equivalent. Who is correct? Explain.

Guided Practice

Write the standard form of the equation of each circle described. Then graph the equation.

6. center at $(0, 0)$, radius 9
7. center at $(-1, 4)$ and tangent to $x = 3$

Write the standard form of each equation. Then graph the equation.

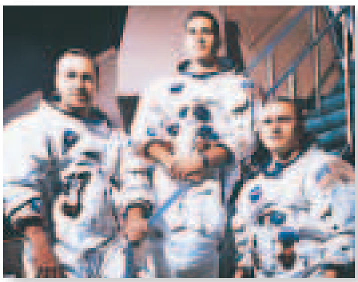
8. $x^2 + y^2 - 4x + 14y - 47 = 0$
9. $2x^2 + 2y^2 - 20x + 8y + 34 = 0$

Write the standard form of the equation of the circle that passes through points with the given coordinates. Then identify the center and radius.

10. $(0, 0), (4, 0), (0, 4)$
11. $(1, 3), (5, 5), (5, 3)$

Write the equation of the circle that satisfies each set of conditions.

12. The circle passes through the point at $(1, 5)$ and has its center at $(-2, 1)$.
13. The endpoints of a diameter are at $(-2, 6)$ and at $(10, -10)$.



14. **Space Science** Apollo 8 was the first manned spacecraft to orbit the moon at an average altitude of 185 kilometers above the moon's surface. Determine an equation to model the orbit of the Apollo 8 command module if the radius of the moon is 1740 kilometers. Let the center of the moon be at the origin.

← Apollo 8 crew: (from left) James A. Lovell, Jr., William A. Anders, Frank Borman

EXERCISES

Practice

Write the standard form of the equation of each circle described. Then graph the equation.

15. center at $(0, 0)$, radius 5
16. center at $(-4, 7)$, radius $\sqrt{3}$
17. center at $(-1, -3)$, radius $\frac{\sqrt{2}}{2}$
18. center at $(-5, 0)$, radius $\frac{9}{2}$
19. center at $(6, 1)$, tangent to the y-axis
20. center at $(3, -2)$, tangent to $y = 2$

Write the standard form of each equation. Then graph the equation.

21. $36 - x^2 = y^2$

22. $x^2 + y^2 + y = \frac{3}{4}$

23. $x^2 + y^2 - 4x + 12y + 30 = 0$

24. $2x^2 + 2y^2 + 2x - 4y = -1$

25. $6x^2 - 12x + 6y^2 + 36y = 36$

26. $16x^2 + 16y^2 - 8x - 32y = 127$

27. Write $x^2 + y^2 + 14x + 24y + 157 = 0$ in standard form. Then graph the equation.

interNET CONNECTION

Graphing Calculator Programs
For a graphing calculator program that determines the radius and the coordinates of the center of a circle from an equation written in general form, visit **www.amc.glencoe.com**

Write the standard form of the equation of the circle that passes through the points with the given coordinates. Then identify the center and radius.

28. $(0, -1), (-3, -2), (-6, -1)$

29. $(7, -1), (11, -5), (3, -5)$

30. $(-2, 7), (-9, 0), (-10, -5)$

31. $(-2, 3), (6, -5), (0, 7)$

32. $(4, 5), (-2, 3), (-4, -3)$

33. $(1, 4), (2, -1), (-3, 0)$

34. Write the standard form of the equation of the circle that passes through the origin and points at $(2.8, 0)$ and $(5, 2)$.

Write the equation of the circle that satisfies each set of conditions.

35. The circle passes through the origin and has its center at $(-4, 3)$.

36. The circle passes through the point $(5, 6)$ and has its center at $(2, 3)$.

37. The endpoints of a diameter are at $(2, 3)$ and at $(-6, -5)$.

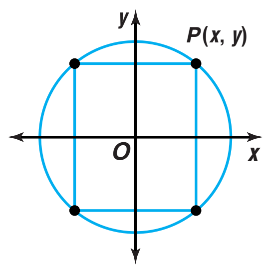
38. The points at $(-3, 4)$ and $(2, 1)$ are the endpoints of a diameter.

39. The circle is tangent to the line with equation $x + 3y = -2$ and has its center at $(5, 1)$.

40. The center of the circle is on the x-axis, its radius is 1, and it passes through the point at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Graphing Calculator

41. A rectangle is inscribed in a circle centered at the origin with diameter 12.



a. Write the equation of the circle that meets these conditions.

b. Write the dimensions of the rectangle in terms of x.

c. Write a function $A(x)$ that represents the area of the rectangle.

d. Use a graphing calculator to graph the function $y = A(x)$.

e. Find the value of x, to the nearest tenth, that maximizes the area of the rectangle. What is the maximum area of the rectangle?

42. Select the standard viewing window and then select **ZSquare** from the ZOOM menu.

a. Select **9:Circle(** from the **DRAW** menu and then enter 2 , 3 , 4) . Then press ENTER.

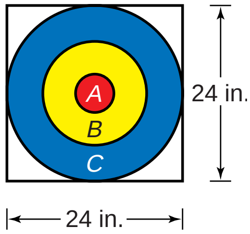
b. Describe what appears on the viewing screen.

c. Write the equation for this graph.

d. Use what you have learned to write the command to graph the equation $(x + 4)^2 + (y - 2) = 36$. Then graph the equation.

Applications and Problem Solving

43. **Sports** Cindy is taking an archery class and decides to practice her skills at home. She attaches the target shown at right to a bale of hay. The circles on the target are concentric and equally spaced apart.



a. If the common center of the circles is located at the origin, write an equation that models the largest circle.

b. If the smallest circle is modeled by the equation $x^2 + y^2 = 6.25$, find the area of the region marked B.

44. **Geometry** Write the equation of a circle that circumscribes the triangle whose sides are the graphs of $4x - 7y = 27$, $x - 5y + 3 = 0$, and $2x + 3y - 7 = 0$.

45. **Critical Thinking** Consider a family of circles in which $h = k$ and the radius is 2. Let k be any real number.

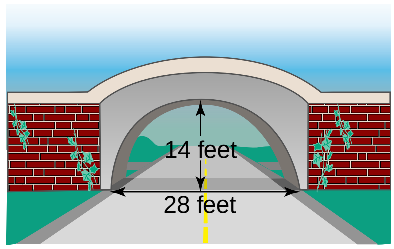
a. Write the equation of the family of circles.

b. Graph three members of this family on the same set of axes.

c. Write a description of all members of this family of circles.

Look Back
Refer to Lesson 3-2 to review families of graphs.

46. **Transportation** A moving truck 7 feet wide and 13 feet high is approaching a semi-circular brick archway at an apartment complex. The base of the archway is 28 feet wide. The road under the archway is divided, allowing for two-way traffic.

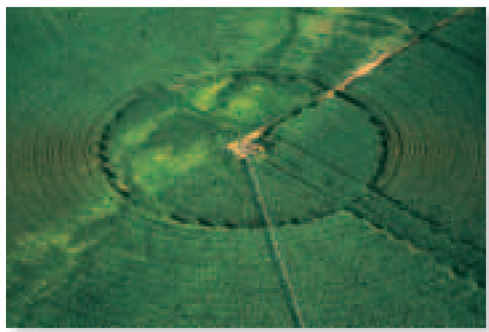


a. Write an equation, centered at the origin, of the archway that models its shape.

b. If the truck remains just to the right of the median, will it be able to pass under the archway without damage? Explain.

47. **Critical Thinking** Find the radius and the coordinates of the center of a circle defined by the equation $x^2 + y^2 - 8x + 6y + 25 = 0$. Describe the graph of this circle.

48. **Agriculture** One method of irrigating crops is called the center pivot system. This system rotates a sprinkler pipe from the center of the field to be irrigated. Suppose a farmer places one of these units at the center of a square plot of land 2500 feet on each side. With the center of this plot at the origin, the irrigator sends out water far enough to reach a point located at (475, 1140).



a. Find an equation representing the farthest points the water can reach.

b. Find the area of the land that receives water directly.

c. About what percent of the farmer's plot does not receive water directly?

49. **Critical Thinking** Consider points $A(3, 4)$, $B(-3, -4)$, and $P(x, y)$.

a. Write an equation for all x and y for which $\overline{PA} \perp \overline{PB}$.

b. What is the relationship between points A, B, and P if $\overline{PA} \perp \overline{PB}$?

Mixed Review

50. Find the distance between points at $(4, -3)$ and $(-2, 6)$. *(Lesson 10-1)*

51. Simplify $(2 + \boldsymbol{i})(3 - 4\boldsymbol{i})(1 + 2\boldsymbol{i})$. *(Lesson 9-5)*

52. Sports Patrick kicked a football with an initial velocity of 60 ft/s at an angle of 60° to the horizontal. After 0.5 seconds, how far has the ball traveled horizontally and vertically. *(Lesson 8-7)*

53. Toys A toy boat floats on the water bobbing up and down. The distance between its highest and lowest point is 5 centimeters. It moves from its highest point down to its lowest point and back up to its highest point every 20 seconds. Write a cosine function that models the movement of the boat in relationship to the equilibrium point. *(Lesson 6-6)*

54. Find the area to the nearest square unit of $\triangle ABC$ if $a = 15$, $b = 25$, and $c = 35$. *(Lesson 5-8)*

55. Amusement Parks The velocity of a roller coaster as it moves down a hill is $\sqrt{v_0^2 + 64h}$. The designer of a coaster wants the coaster to have a velocity of 95 feet per second when it reaches the bottom of the hill. If the initial velocity of the coaster at the top of the hill is 15 feet per second, how high should the designer make the hill? *(Lesson 4-7)*

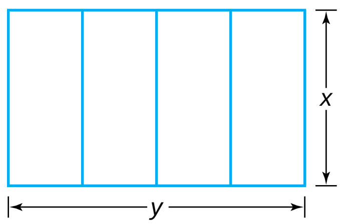
56. Determine whether the graph of $y = 6x^4 - 3x^2 + 1$ is symmetric with respect to the x-axis, the y-axis, the line $y = x$, the line $y = -x$, or the origin. *(Lesson 3-1)*

57. Business A pharmaceutical company manufactures two drugs. Each case of drug A requires 3 hours of processing time and 1 hour of curing time per week. Each case of drug B requires 5 hours of processing time and 5 hours of curing time per week. The schedule allows 55 hours of processing time and 45 hours of curing time weekly. The company must produce no more than 10 cases of drug A and no more than 9 cases of drug B. *(Lesson 2-7)*

a. If the company makes a profit of \$320 on each case of drug A and \$500 on each case of drug B, how many cases of each drug should be produced in order to maximize profit?

b. Find the maximum profit.

58. SAT/ACT Practice An owner plans to divide and enclose a rectangular property with dimensions x and y into rectangular regions as shown at the right. In terms of x and y, what is the total length of fence needed if every line segment represents a section of fence and there is no overlap between sections of fence.



A $5x + 2y$

B $5x + 8y$

C $4x + 2y$

D $4xy$

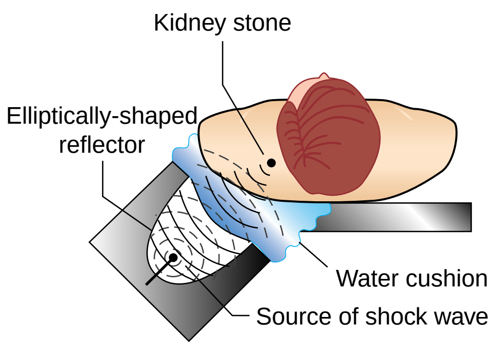
E xy

Extra Practice See p. A44.

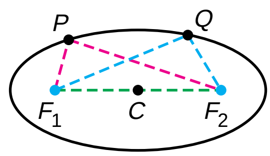
10-3 Ellipses

OBJECTIVES
- Use and determine the standard and general forms of the equation of an ellipse.
- Graph ellipses.

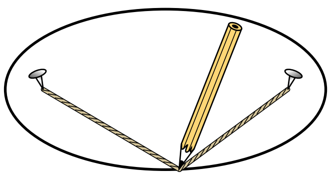
MEDICAL TECHNOLOGY To eliminate kidney stones, doctors sometimes use a medical tool called a *lithotripter,* (*LITH-uh-trip-tor*) which means "stone crusher." A lithotripter is a device that uses ultra-high-frequency shock waves moving through water to break up the stone. After x-raying a patient's kidney to precisely locate and measure the stone, the lithotripter is positioned so that the shock waves reflect off the inner surface of the elliptically-shaped tub and break up the stone. *A problem related to this will be solved in Example 1.*



An **ellipse** is the set of all points in the plane, the sum of whose distances from two fixed points, called **foci,** is constant. In the figure at the right, F_1 and F_2 are the foci, and the midpoint C of the line segment joining the foci is called the **center** of the ellipse. P and Q are any two points on the ellipse. By definition, $PF_1 + PF_2 = QF_1 + QF_2$.

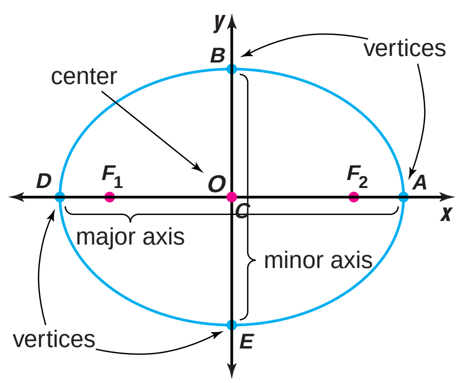


Foci (FOH sigh) is the plural of focus.

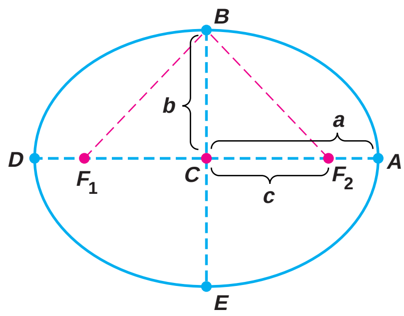


To help visualize this definition, imagine tacking two ends of a string at the foci and using a pencil to trace a curve as it is held tight against the string. The curve which results will be an ellipse since the sum of the distances to the foci, the total length of the string, remains constant.

The parent graph of an ellipse, shown at the right, is centered at the origin. An ellipse has two axes of symmetry, in this case, the x-axis and the y-axis. Notice that the ellipse intersects each axis of symmetry two times. The longer line segment, $\overline{AD}$, which contains the foci, is called the **major axis.** The shorter segment, $\overline{BE}$, is called the **minor axis.** The endpoints of each axis are the **vertices** of the ellipse.



The center separates each axis into two congruent segments. Suppose we let b represent the length of the **semi-minor axis** $\overline{BC}$ and a represent the length of the **semi-major axis** $\overline{CA}$. The foci are located along the major axis, c units from the center. There is a special relationship among the values a, b, and c.



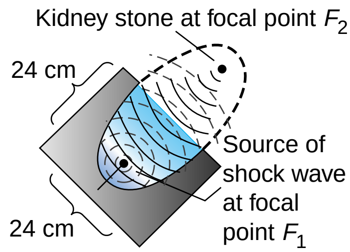
Suppose you draw $\overline{BF_1}$ and $\overline{BF_2}$. The lengths of these segments are equal because $\triangle BF_1C \cong \triangle BF_2C$. Since B and A are two points on the ellipse, you can use the definition of an ellipse to find BF_2.

$BF_1 + BF_2 = AF_1 + AF_2$	*Definition of ellipse*
$BF_1 + BF_2 = AF_1 + DF_1$	$AF_2 = DF_1$
$BF_1 + BF_2 = AD$	*Segment addition:* $AF_1 + DF_1 = AD$
$BF_1 + BF_2 = 2a$	*Substitution:* $AD = 2a$
$2(BF_2) = 2a$	$BF_1 = BF_2$
$BF_2 = a$	

Since $BF_2 = a$ and $\triangle BCF_2$ is a right triangle, $b^2 + c^2 = a^2$ by the Pythagorean Theorem.

Example 1 Real World Application

MEDICAL TECHNOLOGY **Refer to the application at the beginning of this lesson. Suppose the reflector of a mobile lithotripter is 24 centimeters wide and 24 centimeters deep. How far, to the nearest hundredth of a centimeter, should the shock wave emitter be placed from a patient's kidney stone?**



For the lithotripter to break up the stone, the shock wave emitter must be positioned at one focal point of the ellipse and the kidney stone at the other. To determine the distance between the emitter and the kidney stone, we must first find the lengths of the semi-major and semi-minor axes.

The semi-major axis in this ellipse is equal to the depth of the reflector, 24 centimeters. So, $a = 24$.

The semi-minor axis in this ellipse is half the width of the reflector, 12 centimeters. So, $b = 12$.

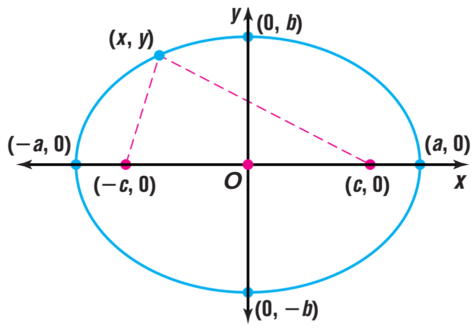
To find the focal length of the ellipse, use the formula $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2$	
$c^2 = (24)^2 - (12)^2$	$a = 24$ *and* $b = 12$
$c^2 = 432$	*Simplify.*
$c = \sqrt{432}$ or approximately 20.8	*Take the square root of each side.*

The distance between the two foci of the ellipse is $2c$ or 41.6.

Therefore, the emitter should be placed approximately 41.6 centimeters away from the patient's kidney along the ellipse's major axis.

The standard form of the equation of an ellipse can be derived from the definition and the distance formula. Consider the special case when the center is at the origin.



Suppose the foci are at $F_1(-c, 0)$ and $F_2(c, 0)$, and (x, y) is any point on the ellipse. By definition, the sum of the distances from a point at (x, y) to the foci is constant. To find a value for this constant, let (x, y) be one of the vertices on the x-axis, for example, $(a, 0)$. Let d_1 and d_2 be the distances from the point at $(a, 0)$ to F_1 and F_2, respectively. Use the Distance Formula.

$$\begin{aligned} d_1 &= \sqrt{[a-(-c)]^2 + (0-0)^2} \\ &= \sqrt{(a+c)^2} \\ &= a + c \end{aligned}$$

$$\begin{aligned} d_2 &= \sqrt{(a-c)^2 + (0-0)^2} \\ &= \sqrt{(a-c)^2} \\ &= a - c \end{aligned}$$

$d_1 + d_2 = a + c + a - c$ or $2a$ Therefore, one value for the constant is $2a$.

Using the Distance Formula and this value for the constant, a general equation for any point at (x, y) is $2a = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}$.

$$2a = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} \quad \textit{Isolate a radical.}$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 \quad \textit{Square each side.}$$

$$a^2 - xc = a\sqrt{(x-c)^2 + y^2} \quad \textit{Simplify.}$$

$$a^4 - 2a^2xc + x^2c^2 = a^2[(x-c)^2 + y^2] \quad \textit{Square each side again.}$$

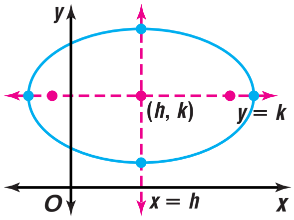
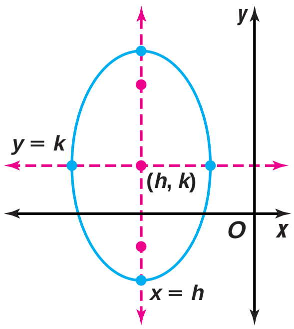
$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2) \quad \textit{Simplify.}$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad \textit{Divide each side by } a^2(a^2 - c^2).$$

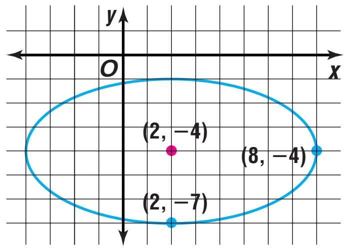
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad b^2 = a^2 - c^2$$

The resulting equation, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is the equation of an ellipse whose center is the origin and whose foci are on the x-axis. When the foci are on the y-axis, the equation is of the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$.

The standard form of the equation of an ellipse with a center other than the origin is a translation of the parent graph to a center at (h, k). The table on the next page gives the standard form, graph, and general description of the equation of each type of ellipse.

Standard Form of the Equation of an Ellipse	Orientation	Description
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $c^2 = a^2 - b^2$		Center: (h, k) Foci: $(h \pm c, k)$ Major axis: $y = k$ Major axis vertices: $(h \pm a, k)$ Minor axis: $x = h$ Minor axis vertices: $(h, k \pm b)$
$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$, where $c^2 = a^2 - b^2$		Center: (h, k) Foci: $(h, k \pm c)$ Major axis: $x = h$ Major axis vertices: $(h, k \pm a)$ Minor axis: $y = k$ Minor axis vertices: $(h \pm b, k)$

Example 2 **Consider the ellipse graphed at the right.**



a. Write the equation of the ellipse in standard form.

b. Find the coordinates of the foci.

a. The center of the graph is at $(2, -4)$. Therefore, $h = 2$ and $k = -4$.

Since the ellipse's horizontal axis is longer than its vertical axis, a is the distance between points at $(2, -4)$ and $(8, -4)$ or 6. The value of b is the distance between points at $(2, -4)$ and $(2, -7)$ or 3.

Therefore, the standard form of the equation of this ellipse is

$$\frac{(x-2)^2}{6^2} + \frac{(y+4)^2}{3^2} = 1 \text{ or } \frac{(x-2)^2}{36} + \frac{(y+4)^2}{9} = 1.$$

b. Using the equation $c = \sqrt{a^2 - b^2}$, we find that $c = 5$. The foci are located on the horizontal axis, 5 units from the center of the ellipse. Therefore, the foci have coordinates $(7, -4)$ and $(-3, -4)$.

In all ellipses, $a^2 \geq b^2$. You can use this information to determine the orientation of the major axis from the equation. If a^2 is the denominator of the x^2 term, the major axis is parallel to the x-axis. If a^2 is the denominator of the y^2 term, the major axis is parallel to the y-axis.

Example 3 For the equation $\frac{(y-3)^2}{25} + \frac{(x+4)^2}{9} = 1$, find the coordinates of the center, foci, and vertices of the ellipse. Then graph the equation.

Determine the values of a, b, c, h, and k.

Since $a^2 > b^2$, $a^2 = 25$ and $b^2 = 9$.

$a^2 = 25$ $\quad$ $b^2 = 9$ $\quad$ $c = \sqrt{a^2 - b^2}$

$a = \sqrt{25}$ or 5 $\quad$ $b = \sqrt{9}$ or 3 $\quad$ $c = \sqrt{25 - 9}$ or 4

$x - h = x + 4$ $\quad$ $y - k = y - 3$

$h = -4$ $\quad$ $k = 3$

Since a^2 is the denominator of the y term, the major axis is parallel to the y-axis.

Graphing Calculator Tip

You can graph an ellipse by first solving for y. Then graph the two resulting equations on the same screen.

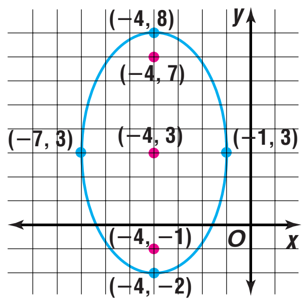
center: $(-4, 3)$ $\quad$ *(h, k)*

foci: $(-4, 7)$ and $(-4, -1)$ $\quad$ *$(h, k \pm c)$*

major axis vertices: $(-4, 8)$ and $(-4, -2)$ $\quad$ *$(h, k \pm a)$*

minor axis vertices: $(-1, 3)$ and $(-7, 3)$ $\quad$ *$(h \pm b, k)$*

Graph these ordered pairs. Other points on the ellipse can be found by substituting values for x and y. Complete the ellipse.



As with circles, the standard form of the equation of an ellipse can be expanded to obtain the general form. The result is a second-degree equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A \neq 0$ and $C \neq 0$, and A and C have the same sign. An equation in general form can be rewritten in standard form to determine the center at (h, k), the measure of the semi-major axis, a, and the measure of the semi-minor axis, b.

Example 4 Find the coordinates of the center, the foci, and the vertices of the ellipse with the equation $4x^2 + 9y^2 - 40x + 36y + 100 = 0$. Then graph the equation.

First write the equation in standard form.

$$4x^2 + 9y^2 - 40x + 36y + 100 = 0$$

$$4(x^2 - 10x + ?) + 9(y^2 + 4y + ?) = -100 + ? + ?$$ *The GCF of the x terms is 4. The GCF of the y terms is 9.*

$$4(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = -100 + 4(25) + 9(4)$$ *Complete the square.*

$$4(x - 5)^2 + 9(y + 2)^2 = 36$$ *Factor.*

$$\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} = 1$$ *Divide each side by 36.*

(continued on the next page)

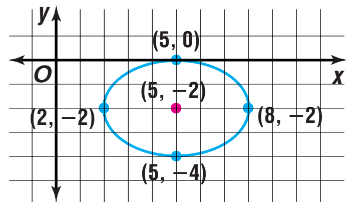
Since $a^2 > b^2$, $a^2 = 9$ and $b^2 = 4$. Thus, $a = 3$ and $b = 2$.

Since $c^2 = a^2 - b^2$, $c = \sqrt{5}$.

Since a^2 is the denominator of the x term, the major axis is parallel to the x-axis.

center: $(5, -2)$ *(h, k)*
foci: $\left(5 \pm \sqrt{5}, -2\right)$ *$(h \pm c, k)$*
major axis vertices: $(8, -2)$ and $(2, -2)$ *$(h \pm a, k)$*
minor axis vertices: $(5, 0)$ and $(5, -4)$ *$(h, k \pm b)$*

Sketch the ellipse.



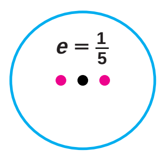
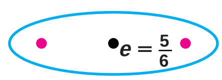
The **eccentricity** of an ellipse, denoted by e, is a measure that describes the shape of an ellipse. It is defined as $e = \frac{c}{a}$. Since $0 < c < a$, you can divide by a to show that $0 < e < 1$.

$0 < c < a$

$0 < \frac{c}{a} < 1$ *Divide by a.*

$0 < e < 1$ *Replace $\frac{c}{a}$ with e.*

The table shows the relationship between the value of e, the location of the foci, and the shape of the ellipse.

Value of e	Location of Foci	Graph
close to 0	near center of ellipse	
close to 1	far from center of ellipse	

Sometimes, you may need to find the value of b when you know the values of a and e. In any ellipse, $b^2 = a^2 - c^2$ and $\frac{c}{a} = e$. By using the two equations, it can be shown that $b^2 = a^2(1 - e^2)$. *You will derive this formula in Exercise 4.*

All of the planets in our solar system have elliptical orbits with the sun as one focus. These orbits are often described by their eccentricity.

Example 5 Real World Application

ASTRONOMY **Of the nine planetary orbits in our solar system, Pluto's has the greatest eccentricity, 0.248. Astronomers have determined that the orbit is about 29.646 AU (astronomical units) from the sun at its closest point to the sun (perihelion). The length of the semi-major axis is about 39.482 AU.**

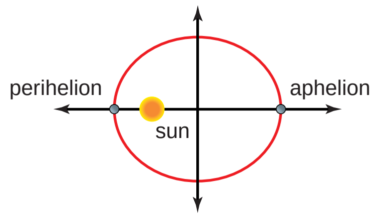
1 AU = the average distance between the sun and Earth, about 9.3×10^7 miles

a. Sketch the orbit of Pluto showing the sun in its position.

b. Find the length of the semi-minor axis of the orbit.

c. Find the distance of Pluto from the sun at its farthest point (aphelion).

a. The sketch at the right shows the sun as a focus for the elliptical orbit.



b. b is the length of the semi-minor axis.

$$b^2 = a^2(1 - e^2)$$

$$b = \sqrt{a^2(1 - e^2)}$$

$$b = \sqrt{(39.482)^2(1 - 0.248^2)} \quad \textit{a = 39.482, e = 0.248}$$

$$b \approx 38.249 \text{ AU}$$

c. *aphelion = length of major axis − distance from sun to perihelion*

$$d = 2(39.482) - 29.646$$

$$= 49.318$$

Pluto is about 49 AU from the sun at its aphelion.

CHECK FOR UNDERSTANDING

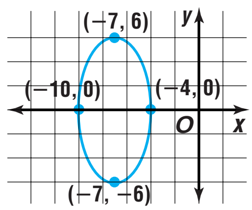
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** the equation of an ellipse centered at the origin, with $a = 8$, $b = 5$, and the major axis on the y-axis.
2. **Explain** how to determine whether the foci of an ellipse lie on the horizontal or vertical axis of an ellipse.
3. **Describe** the result when the foci and center of an ellipse coincide and give the eccentricity of such an ellipse.
4. **Derive** the equation $b^2 = a^2(1 - e^2)$.
5. **You Decide** Manuel says that the graph of $3y - 36 = 2x^2 - 18x$ is an ellipse with its major axis parallel to the y-axis. Shanice disagrees. Who is correct? Explain your answer.

Guided Practice

6. Write the equation of the ellipse graphed at the right in standard form. Then find the coordinates of its foci.



For the equation of each ellipse, find the coordinates of the center, foci, and vertices. Then graph the equation.

7. $\frac{x^2}{36} + \frac{y^2}{4} = 1$

8. $\frac{x^2}{81} + \frac{(y - 4)^2}{49} = 1$

9. $25x^2 + 9y^2 + 100x - 18y = 116$

10. $9x^2 + 4y^2 - 18x + 16y = 11$

Write the equation of the ellipse that meets each set of conditions.

11. The center is at $(-2, -3)$, the length of the vertical major axis is 8 units, and the length of the minor axis is 2 units.

12. The foci are located at $(-1, 0)$ and $(1, 0)$ and $a = 4$.

13. The center is at $(1, 2)$, the major axis is parallel to the x-axis, and the ellipse passes through points at $(1, 4)$ and $(5, 2)$.

14. The center is at $(3, 1)$, the vertical semi-major axis is 6 units long, and $e = \frac{1}{3}$.

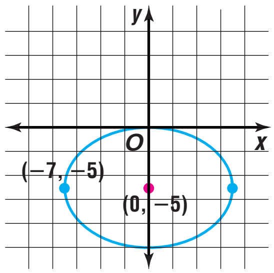
15. **Astronomy** The elliptical orbit of Mars has its foci at $(0.141732, 0)$ and $(-0.141732, 0)$, where 1 unit equals 1 AU. The length of the major axis is 3.048 AU. Determine the equation that models Mars' elliptical orbit.

EXERCISES

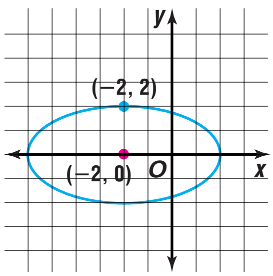
Practice

Write the equation of each ellipse in standard form. Then find the coordinates of its foci.

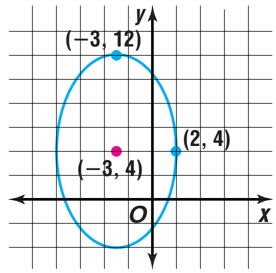
16.



17.



18.



For the equation of each ellipse, find the coordinates of the center, foci, and vertices. Then graph the equation.

19. $\frac{(x+2)^2}{1} + \frac{(y-1)^2}{4} = 1$

20. $\frac{(x-6)^2}{100} + \frac{(y-7)^2}{121} = 1$

21. $\frac{(x-4)^2}{16} + \frac{(y+6)^2}{9} = 1$

22. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

23. $4x^2 + y^2 - 8x + 6y + 9 = 0$

24. $16x^2 + 25y^2 - 96x - 200y = -144$

25. $3x^2 + y^2 + 18x - 2y + 4 = 0$

26. $6x^2 - 12x + 6y^2 + 36y = 36$

27. $18y^2 + 12x^2 - 144y - 48x = -120$

28. $4y^2 - 8y + 9x^2 - 54x + 49 = 0$

29. $49x^2 + 16y^2 + 160y - 384 = 0$

30. $9y^2 + 108y + 4x^2 - 56x = -484$

Write the equation of the ellipse that meets each set of conditions.

31. The center is at $(-3, -1)$, the length of the horizontal semi-major axis is 7 units, and the length of the semi-minor axis is 5 units.

32. The foci are at $(-2, 0)$ and $(2, 0)$, and $a = 7$.

33. The length of the semi-minor axis is $\frac{3}{4}$ the length of the horizontal semi-major axis, the center is at the origin, and $b = 6$.

34. The semi-major axis has length $2\sqrt{13}$ units, and the foci are at $(-1, 1)$ and $(-1, -5)$.

35. The endpoints of the major axis are at $(-11, 5)$ and $(7, 5)$. The endpoints of the minor axis are at $(-2, 9)$ and $(-2, 1)$.

36. The foci are at $(1, -1)$ and $(1, 5)$, and the ellipse passes through the point at $(4, 2)$.

37. The center is the origin, $\frac{1}{2} = \frac{c}{a}$, and the length of the horizontal semi-major axis is 10 units.

38. The ellipse is tangent to the x- and y-axes and has its center at $(4, -7)$.

39. The ellipse has its center at the origin, $a = 2$, and $e = \frac{3}{4}$.

40. The foci are at $(3, 5)$ and $(1, 5)$, and the ellipse has eccentricity 0.25.

41. The ellipse has a vertical major axis of 20 units, its center is at $(3, 0)$, and $e = \frac{7}{10}$.

42. The center is at $(1, -1)$, one focus is located at $\left(1, -1 + \sqrt{5}\right)$, and the ellipse has eccentricity $\frac{\sqrt{5}}{3}$.

Graphing Calculator

Graph each equation. Then use the TRACE function to approximate the coordinates of the vertices to the nearest integer.

43. $x^2 + 4y^2 - 6x + 24y = -41$

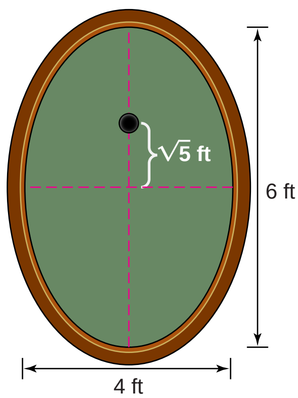
44. $4x^2 + y^2 - 8x - 2y = -1$

45. $4x^2 + 9y^2 - 16x + 18y = 11$

46. $25y^2 + 16x^2 - 150y + 32x = 159$

Applications and Problem Solving

47. Entertainment *Elliptipool* is a billiards game that use an elliptically-shaped pool table with only one pocket in the surface. A cue ball and a target ball are used in play. The object of the game is to strike the target ball with the cue ball so that the target ball rolls into the pocket after one bounce off the side. Suppose the cue ball and target ball can be placed anywhere on the half of the table opposite the pocket. The pool table shown at the right is 4 feet wide and 6 feet long. The pocket is located $\sqrt{5}$ feet from the center of the table along the ellipse's major axis. Assuming no spin is placed on either ball and the target ball is struck squarely, where should each be placed to have the best chance of hitting the target ball into the pocket? Explain your reasoning.



48. Critical Thinking As the foci of an ellipse move farther apart with the major axis fixed, what figure does the ellipse approach? Justify your answer.

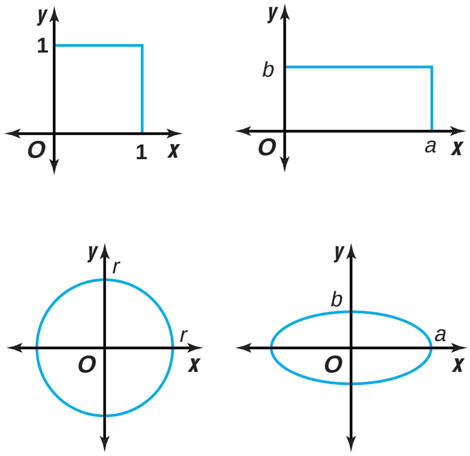
49. History A whispering gallery is designed using an elliptical ceiling. It operates on the principle that sound projected from one focus of an ellipse reflects off the ceiling and back to the other focus. The United States Capitol contains such an elliptical room. The room is 96 feet in length, 46 feet in width, and has a ceiling that is about 23 feet high.

a. Write an equation modeling the shape of the room. Assume that is centered at the origin and the major axis is horizontal.

b. John Quincy Adams is known to have overheard conversations being held at the opposing party leader's desk by standing in a certain spot in this room. Describe two possible places where Adams might have stood to overhear.

c. About how far did Adams stand from the desk?

50. Geometry The square has an area of 1 square unit. If the square is stretched horizontally by a factor of a and compressed vertically by a factor of b, the area of the rectangle formed is ab square units.

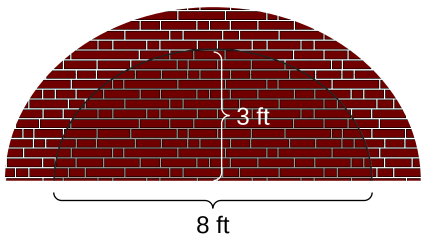


a. The area of a circle with equation $x^2 + y^2 = r^2$ is πr^2. Develop a formula for the area of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

b. Use the formula found in part **a** to find the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

51. Critical Thinking Show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetric with respect to the origin.

52. Construction The arch of a fireplace is to be constructed in the shape of a semi-ellipse. The opening is to have a height of 3 feet at the center and a width of 8 feet along the base. To sketch the outline of the fireplace, the contractor uses an 8-foot string tied to two thumbtacks.



a. Where should the thumbtacks be placed?

b. Explain why this technique works.

53. Astronomy The satellites orbiting Earth follow elliptical paths with the center of Earth as one focus. The table below lists data on five satellites that have orbited or currently orbit Earth.

Satellites Orbiting Earth

Name	Launch Date	Time or expected time aloft	Semi-major axis a (km)	Eccentricity
Sputnik I	Oct. 1957	57.6 days	6955	0.052
Vanguard I	Mar. 1958	300 years	8872	0.208
Skylab 4	Nov. 1973	84.06 days	6808	0.001
GOES 4	Sept. 1980	10^6 years	42,166	0.0003
Intelsat 5	Dec. 1980	10^6 years	41,803	0.007

a. Which satellite has the most circular orbit? Explain your reasoning.

b. Soviet satellite Sputnik I was the first artificial satellite to orbit Earth. If the radius of Earth is approximately 6357 kilometers, find the greatest distance Sputnik I orbited from the surface of Earth to the nearest kilometer.

Mixed Review

54. Write the standard form of the equation of the circle that passes through points at (0, −9), (7, −2), and (−5, −10). Identify the circle's center and radius. Then graph the equation. *(Lesson 10-2)*

55. Determine whether the quadrilateral with vertices at points (−1, −2), (5, −4), (4, 1), and (−5, 4) is a parallelogram. *(Lesson 10-1)*

56. If $\sin \theta = \frac{7}{8}$ and the terminal side of θ is in the first quadrant, find $\cos 2\theta$. *(Lesson 7-4)*

57. Write an equation of the cosine function with an amplitude of 4, a period of 180°, and a phase shift of 20°. *(Lesson 6-5)*

58. Solve $\triangle ABC$ if $C = 121°32'$, $B = 42°5'$, and $a = 4.1$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-6)*

59. Use the Remainder Theorem to find the remainder for the quotient $(x^4 - 4x^3 - 2x^2 - 1) \div (x - 5)$. Then, state whether the binomial is a factor of the polynomial. *(Lesson 4-3)*

60. Determine whether the point at $(-2, -16)$ is the location of a *minimum,* a *maximum,* or a *point of inflection* for the function $x^2 + 4x - 12$. *(Lesson 3-6)*

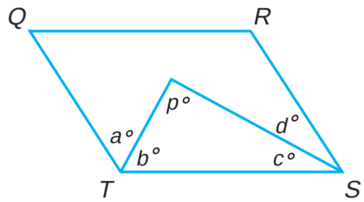
61. Sketch the graph of $g(x) = |x - 2|$. *(Lesson 3-2)*

62. **Motion** A ceiling fan has four evenly spaced blades that are each 2 feet long. Suppose the center of the fan is located at the origin and the blades of the fan lie in the x- and y-axis. Imagine a fly lands on the tip of the blade along the positive x-axis. Find the location of the fly where it landed and its location after a 90°, 180°, and 270° counterclockwise rotation of the fan. *(Lesson 2-4)*

63. **SAT Practice** In parallelogram $QRST$, $a = b$ and $c = d$. What is the value of p?

A 45 **B** 60 **C** 90 **D** 120

E Cannot be determined from the information given



GRAPHING CALCULATOR EXPLORATION

To graph an ellipse, such as the equation $\frac{(x-3)^2}{18} + \frac{(y+2)^2}{32} = 1$, on a graphing calculator, you must first solve for y.

$$\frac{(x-3)^2}{18} + \frac{(y+2)^2}{32} = 1$$

$$32(x-3)^2 + 18(y-2)^2 = 576$$

$$18(y-2)^2 = 576 - 32(x-3)^2$$

$$(y-2)^2 = \frac{576 - 32(x-3)^2}{18}$$

So, $y = \pm\sqrt{\frac{576 - 32(x-3)^2}{18}} + 2$.

The result is two equations. To graph both equations as **Y1**, replace $\pm$ with $\{1, -1\}$.

Like other graphs, there are families of ellipses. Changing certain values in the equation of an ellipse creates a new member of that family.

TRY THESE **For each situation, make a conjecture about the behavior of the graph. Then verify by graphing the original equation and the modified equation on the same screen using a square window.**

1. x is replaced by $(x - 4)$.
2. x is replaced by $(x + 4)$.
3. y is replaced by $(y - 4)$.
4. y is replaced by $(y + 4)$.
5. 32 is switched with 18.

WHAT DO YOU THINK?

6. Describe the effects of replacing x in the equation of an ellipse with $(x \pm c)$ for $c > 0$.
7. Describe the effects of replacing y in the equation of an ellipse with $(y \pm c)$ for $c > 0$.
8. In the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, describe the effect of interchanging a and b.

Extra Practice See p. A44.

10-4

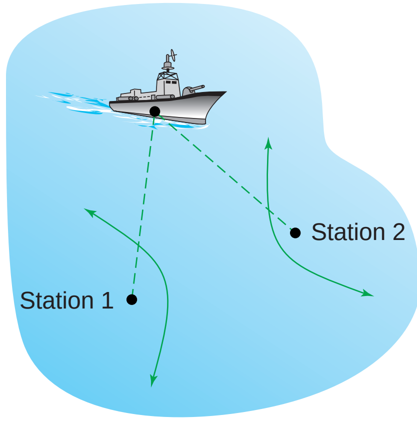
Hyperbolas

OBJECTIVES

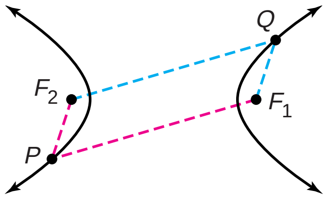
- Use and determine the standard and general forms of the equation of a hyperbola.
- Graph hyperbolas.

Real World Application

NAVIGATION Since World War II, ships have used the LORAN (*LO*ng *RA*nge *N*avigation) system as a means of navigation independent of visibility conditions. Two stations, located a great distance apart, simultaneously transmit radio pulses to ships at sea. Since a ship is usually closer to one station than the other, the ship receives these pulses at slightly different times. By measuring the time differential and by knowing the speed of the radio waves, a ship can be located on a conic whose foci are the positions of the two stations. *A problem related to this will be solved in Example 4.*



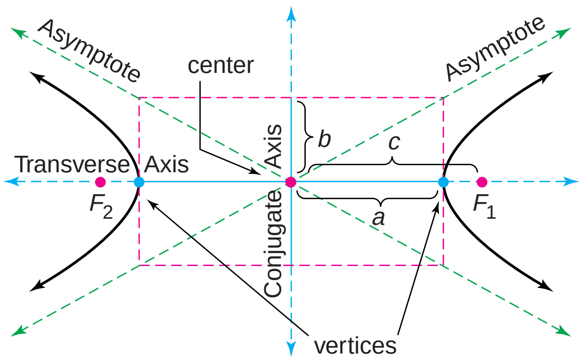
A **hyperbola** is the set of all points in the plane in which the difference of the distances from two distinct fixed points, called **foci,** is constant. That is, if F_1 and F_2 are the foci of a hyperbola and P and Q are any two points on the hyperbola, $|PF_1 - PF_2| = |QF_1 - QF_2|$.



The **center** of a hyperbola is the midpoint of the line segment whose endpoints are the foci. The point on each branch of the hyperbola that is nearest the center is called a **vertex.**

Look Back
Refer to Lesson 3-7 to review asymptotes.

The **asymptotes** of a hyperbola are lines that the curve approaches as it recedes from the center. As you move farther out along the branches, the distance between points on the hyperbola and the asymptotes approaches zero.

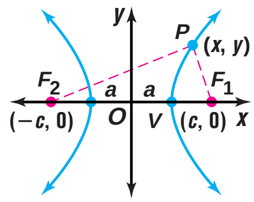


Note that $c > a$ for the hyperbola.

A hyperbola has two axes of symmetry. The line segment connecting the vertices is called the **transverse axis** and has a length of $2a$ units. The segment perpendicular to the transverse axis through the center is called the **conjugate axis** and has length $2b$ units.

For a hyperbola, the relationship among a, b, and c is represented by $a^2 + b^2 = c^2$. The asymptotes contain the diagonals of the rectangle guide, which is $2a$ units by $2b$ units. The point at which the diagonals meet coincides with the center of the hyperbola.

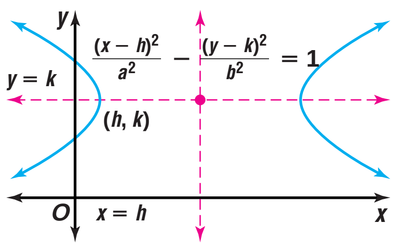
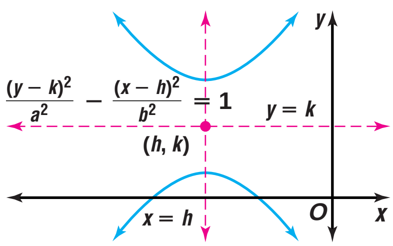
The standard form of the equation of a hyperbola with its origin as its center can be derived from the definition and the Distance Formula. Suppose the foci are on the x-axis at $(c, 0)$ and $(-c, 0)$ and the coordinates of any point on the hyperbola are (x, y).



$\lvert PF_2 - PF_1 \rvert = \lvert VF_2 - VF_1 \rvert$	*Definition of hyperbola*
$\left\lvert \sqrt{(x + c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} \right\rvert = \lvert c + a - (c - a) \rvert$	*Distance Formula*
$\sqrt{(x + c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = 2a$	*Simplify.*
$\sqrt{(x - c)^2 + y^2} = 2a + \sqrt{(x + c)^2 + y^2}$	*Isolate a radical.*
$(x - c)^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$	*Square each side.*
$-4xc - 4a^2 = 4a\sqrt{(x + c)^2 + y^2}$	*Simplify.*
$xc + a^2 = -a\sqrt{(x + c)^2 + y^2}$	*Divide each side by −4.*
$x^2c^2 + 2a^2xc + a^4 = a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$	*Square each side.*
$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$	*Simplify.*
$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$	*Divide by $a^2(c^2 - a^2)$.*
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	*By the Pythagorean Theorem, $c^2 - a^2 = b^2$.*

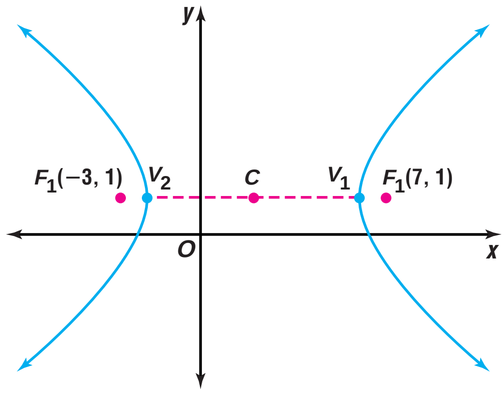
If the foci are on the y-axis, the equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

As with the other graphs we have studied in this chapter, the standard form of the equation of a hyperbola with center other than the origin is a translation of the parent graph to a center at (h, k).

Standard Form of the Equation of a Hyperbola	Orientation	Description
$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$, where $b^2 = c^2 - a^2$		center: (h, k) foci: $(h \pm c, k)$ vertices: $(h \pm a, k)$ equation of transverse axis: $y = k$ *(parallel to x-axis)*
$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$, where $b^2 = c^2 - a^2$		center: (h, k) foci: $(h, k \pm c)$ vertices: $(h, k \pm a)$ equation of transverse axis: $x = h$ *(parallel to y-axis)*

Example 1 **Find the equation of the hyperbola with foci at (7, 1) and (−3, 1) whose transverse axis is 8 units long.**

A sketch of the graph is helpful. Let F_1 and F_2 be the foci, let V_1 and V_2 be the vertices, and let C be the center.



To locate the center, find the midpoint of $\overline{F_1F_2}$.

$$\left(\frac{7+(-3)}{2}, \frac{1+1}{2}\right) \text{ or } (2, 1)$$

Thus, $h = 2$ and $k = 1$ since (2, 1) is the center.

The transverse axis is 8 units long. Thus, $2a = 8$ or $a = 4$. So $a^2 = 16$.

Use the equation $b^2 = c^2 - a^2$ to find b^2.

$b^2 = c^2 - a^2$

$b^2 = 25 - 16$ $\quad$ *$c = 5, a = 4$*

$b^2 = 9$

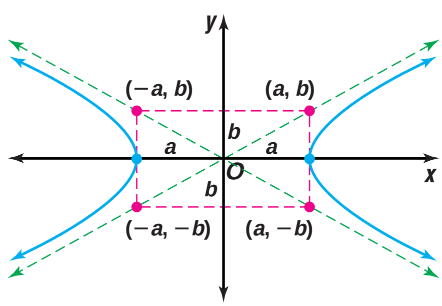
Recall that c is the distance from the center to a focus. Here $c = CF_1$ or 5.

Use the standard form when the transverse axis is parallel to the x-axis.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \rightarrow \frac{(x-2)^2}{16} - \frac{(y-1)^2}{9} = 1$$

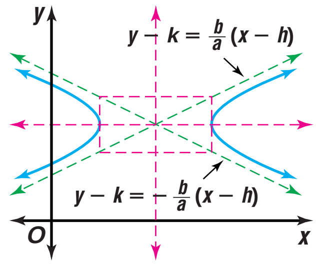
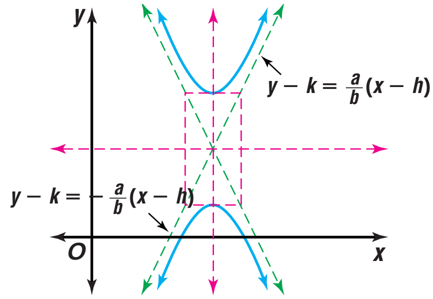
Before graphing a hyperbola, it is often helpful to graph the asymptotes. As noted in the beginning of the lesson, the asymptotes contain the diagonals of the rectangle guide defined by the transverse and conjugate axes. While not part of the graph, a sketch of this $2a$ by $2b$ rectangle provides an easy way to graph the asymptotes of the hyperbola. Suppose the center of a hyperbola is the origin and the transverse axis lies along the x-axis.

From the figure at the right, we can see that the asymptotes have slopes equal to $\pm\frac{b}{a}$. Since both lines have a y-intercept of 0, the equations for the asymptotes are $y = \pm\frac{b}{a}x$.



If the hyperbola were oriented so that the traverse axis was parallel to the y-axis, the slopes of the asymptotes would be $\pm\frac{a}{b}$. Thus, the equations of the asymptotes would be $y = \pm\frac{a}{b}x$.

The equations of the asymptotes of any hyperbola can be determined by a translation of the graph to a center at (h, k).

Equations of the Asymptotes of a Hyperbola	$y - k = \pm\frac{b}{a}(x - h)$, for a hyperbola with standard form $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$	
	$y - k = \pm\frac{a}{b}(x - h)$, for a hyperbola with standard form $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$	

Example 2 **Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $\frac{(y + 4)^2}{36} - \frac{(x - 2)^2}{25} = 1$. Then graph the equation.**

Since the y terms are in the first expression, the hyperbola has a vertical transverse axis.

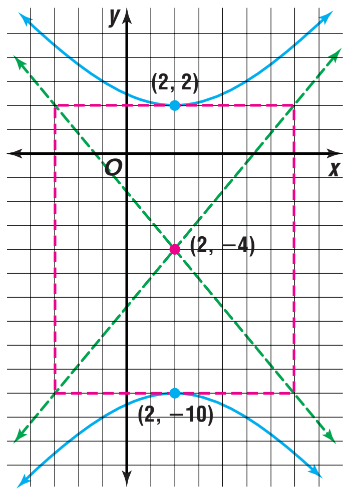
From the equation, $h = 2$, $k = -4$, $a = 6$, and $b = 5$. The center is at $(2, -4)$.

The equations of the asymptotes are $y + 4 = \pm\frac{6}{5}(x - 2)$.

The vertices are at $(h, k \pm a)$ or $(2, 2)$ and $(2, -10)$.

Since $c^2 = a^2 + b^2$, $c = \sqrt{61}$. Thus, the foci are at $\left(2, -4 + \sqrt{61}\right)$ and $\left(2, -4 - \sqrt{61}\right)$.

Graph the center, vertices, and the rectangle guide. Next graph the asymptotes. Then sketch the hyperbola.



Graphing Calculator Tip

You can graph a hyperbola on a graphing calculator by first solving for y and then graphing the two resulting equations on the same screen.

By expanding the standard form for a hyperbola, you can determine the general form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ where $A \neq 0$, $C \neq 0$, and A and C have different signs.

As with other general forms we have studied, the general form of a hyperbola can be rewritten in standard form. While it is important to be able to recognize the equation of a hyperbola in general form, the standard form provides important information about the hyperbola that makes it easier to graph.

Example 3 **Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $9x^2 - 4y^2 - 54x - 40y - 55 = 0$. Then graph the equation.**

Write the equation in standard form. Use the same process you used with ellipses.

$$9x^2 - 4y^2 - 54x - 40y - 55 = 0$$

$$9x^2 - 54x - 4y^2 - 40y = 55 \quad \textit{Rearrange terms.}$$

$$9(x^2 - 6x + ?) - 4(y^2 + 10y + ?) = 55 + ? + ? \quad \textit{Factor GCF for each variable.}$$

$$9(x^2 - 6x + 9) - 4(y^2 + 10y + 25) = 55 + 9(9) - 4(25) \quad \textit{Complete the square.}$$

$$9(x - 3)^2 - 4(y + 5)^2 = 36 \quad \textit{Factor.}$$

$$\frac{(x-3)^2}{4} - \frac{(y+5)^2}{9} = 1 \quad \textit{Divide each side by 36.}$$

The center is at $(3, -5)$. Since the x terms are in the first expression, the transverse axis is horizontal.

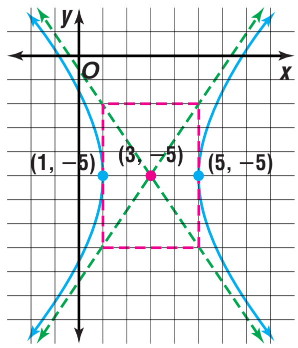
$a = 2$, $b = 3$, $c = \sqrt{13}$

The foci are at $\left(3 - \sqrt{13}, -5\right)$ and $\left(3 + \sqrt{13}, -5\right)$.

The vertices are at $(1, -5)$ and $(5, -5)$.

The asymptotes have equations $y + 5 = \pm\frac{3}{2}(x - 3)$.

Graph the vertices and the rectangle guide. Next graph the asymptotes. Then sketch the hyperbola.



One interesting application of hyperbolas is in navigation.

Example 4 Real World Application

NAVIGATION **Refer to the application at the beginning of the lesson. Suppose LORAN stations *A* and *B* are located 400 miles apart along a straight shore, with *A* due west of *B*. A ship approaching the shore receives radio pulses from the stations and is able to determine that it is 100 miles farther from station *A* than it is from station *B*.**

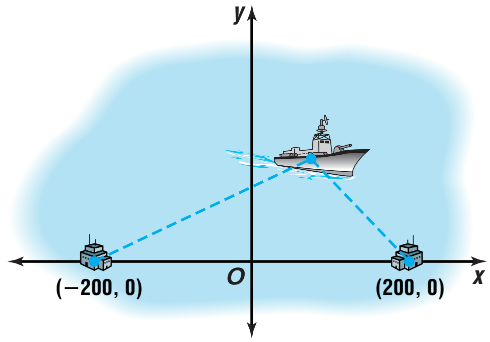
a. Find the equation of the hyperbola on which the ship is located.

b. Find the exact coordinates of the ship if it is 60 miles from shore.

a. First set up a rectangular coordinate system with the origin located midway between station *A* and station *B*.

The stations are located at the foci of the hyperbola, so $c = 200$.

The difference of the distances from the ship to each station is 100 miles. By definition of a hyperbola, this difference equals $2a$, so $a = 50$. The vertices of the hyperbola are located on the same axis as the foci, so the vertices of the hyperbola the ship is on are at $(-50, 0)$ and $(50, 0)$.



Since the hyperbola's transverse axis is the x-axis, the form of the equation of this hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Using the equation $b^2 = c^2 - a^2$, we can find the value for b^2.

$$b^2 = c^2 - a^2$$
$$b^2 = 200^2 - 50^2 \quad a = 50,\ c = 200$$
$$b^2 = 37{,}500$$

Thus, the equation of the hyperbola is $\frac{x^2}{2500} - \frac{y^2}{37{,}500} = 1$

b. If the ship is 60 miles from shore, let $y = 60$ in the equation of the hyperbola and solve for x.

$$\frac{x^2}{2500} - \frac{60^2}{37{,}500} = 1$$
$$\frac{x^2}{2500} = 1 + \frac{60^2}{37{,}500}$$
$$\frac{x^2}{2500} = 1.096$$
$$x^2 = 2500(1.096)$$
$$x = \pm\sqrt{2740} \approx \pm 52.3$$

Since the ship is closer to station B than station A, we use the positive value of x to locate the ship at coordinates $(52.3, 60)$.

In the standard form of the equation of a hyperbola, if $a = b$, the graph is an **equilateral hyperbola.** Replacing a with b in the equations of the asymptotes of a hyperbola with a horizontal transverse axis reveals a property of equilateral hyperbolas.

$$y - k = \frac{b}{a}(x - h) \qquad y - k = -\frac{b}{a}(x - h)$$
$$y - k = \frac{b}{b}(x - h) \quad \text{Let } a = b \quad y - k = -\frac{b}{b}(x - h)$$
$$y - k = (x - h) \qquad y - k = -(x - h)$$

The slopes of the equations of the two asymptotes are negative reciprocals, 1 and -1. Thus, the asymptotes of an equilateral hyperbola are perpendicular.

Remember that $xy = c$ is the general equation that models inverse variation.

A special case of the equilateral hyperbola is a **rectangular hyperbola,** where the coordinate axes are the asymptotes. The general equation of a rectangular hyperbola is $xy = c$, where c is a nonzero constant. The sign of the constant c determines the location of the branches of the hyperbola.

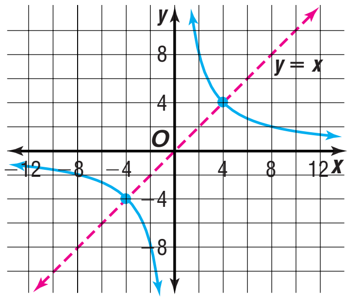
Rectangular Hyperbola: $xy = c$	
Value of c	**Location of branches of hyperbola**
Positive	Quadrants I and III
Negative	Quadrants II and IV

Example 5 **Graph $xy = 16$.**

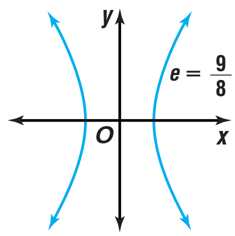
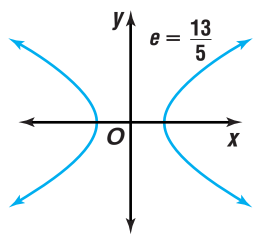
Since c is positive, the hyperbola lies in the first and third quadrants.

The transverse axis is along the graph of $y = x$.

The coordinates of the vertices must satisfy the equation of the hyperbola and also their graph must be points on the traverse axis. Thus, the vertices are at (4, 4) and (−4, −4).



Like an ellipse, the shape of a hyperbola is determined by its eccentricity, which is again defined as $e = \frac{c}{a}$. However, in a hyperbola, $0 < a < c$. So, $0 < 1 < e$ or $e > 1$. The table below shows the relationship between the value of the e and the shape of the hyperbola.

Value of e	**Graph**
close to 1	
not close to 1	

Since $c^2 = a^2 + b^2$ in hyperbolas, it can be shown that $b^2 = a^2(e^2 - 1)$. *You will derive this formula in Exercise 3.*

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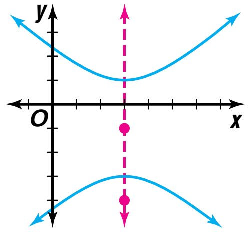
Example 6 **Write the equation of the hyperbola with center at (3, −1), a focus at (3, −4), and eccentricity $\frac{3}{2}$.**

Sketch the graph using the points given. Since the center and focus have the same x-coordinate, the transverse axis is vertical. Use the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.



The focus is 3 units below the center, so $c = 3$. Now use the eccentricity to find the values of a^2 and b^2.

$e = \frac{c}{a}$

$\frac{3}{2} = \frac{3}{a}$ *$c = 3$ and $e = \frac{3}{2}$*

$2 = a$

$4 = a^2$

$b^2 = a^2(e^2 - 1)$

$b^2 = 4\left(\frac{9}{4} - \frac{4}{4}\right)$ *$a^2 = 4$ and $e = \frac{3}{2}$*

$b^2 = 5$

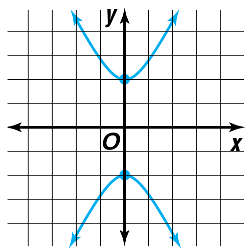
The equation is $\frac{(y+1)^2}{4} - \frac{(x-3)^2}{5} = 1$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** the standard forms of the equations of hyperbolas and ellipses.
2. **Determine** which of the following equations matches the graph of the hyperbola at right.



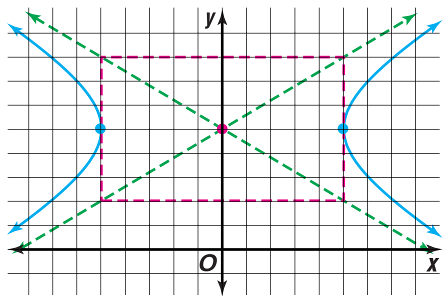
   a. $\frac{x^2}{4} - y^2 = 1$ b. $\frac{y^2}{4} - x^2 = 1$ c. $x^2 - \frac{y^2}{4} = 1$
3. **Derive** the equation $b^2 = a^2(e^2 - 1)$ for a hyperbola.
4. *Math Journal* **Write** an explanation of how to determine whether the transverse axis of a hyperbola is horizontal or vertical.

Guided Practice

For the equation of each hyperbola, find the coordinates of the center, the foci, and the vertices and the equations of the asymptotes of its graph. Then graph the equation.

5. $\frac{x^2}{25} - \frac{y^2}{4} = 1$ 6. $\frac{(y-3)^2}{16} - \frac{(x-2)^2}{4} = 1$ 7. $y^2 - 5x^2 + 20x = 50$

8. Write the equation of the hyperbola graphed at the right.



9. Graph $xy = -9$.

Write an equation of the hyperbola that meets each set of conditions.

10. The center is at $(1, -4)$, $a = 5$, $b = 2$, and it has a horizontal transverse axis.

11. The length of the conjugate axis is 6 units, and the vertices are at $(3, 4)$ and $(3, 0)$.

12. The hyperbola is equilateral and has foci at $(0, 6)$ and $(0, -6)$.

13. The eccentricity of the hyperbola is $\frac{5}{3}$, and the foci are at $(10, 0)$ and $(-10, 0)$.

14. Aviation Airplanes are equipped with signal devices to alert rescuers to their position. Suppose a downed plane sends out radio pulses that are detected by two receiving stations, A and B. The stations are located 130 miles apart along a stretch of I-40, with A due west of B. The two stations are able to determine that the plane is 50 miles farther from station B than from station A.

a. Determine the equation of the hyperbola centered at the origin on which the plane is located.

b. Graph the equation, indicating on which branch of the hyperbola the plane is located.

c. If the pilot estimates that the plane is 6 miles from I-40, find the exact coordinates of its position.

EXERCISES

Practice

For the equation of each hyperbola, find the coordinates of the center, the foci, and the vertices and the equations of the asymptotes of its graph. Then graph the equation.

15. $\frac{x^2}{100} - \frac{y^2}{16} = 1$

16. $\frac{x^2}{9} - \frac{(y-5)^2}{81} = 1$

17. $\frac{x^2}{4} - \frac{y^2}{49} = 1$

18. $\frac{(y-7)^2}{64} - \frac{(x+1)^2}{4} = 1$

19. $x^2 - 4y^2 + 6x - 8y = 11$

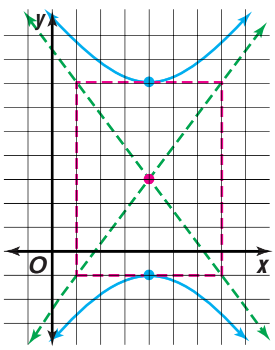
20. $-4x^2 + 9y^2 - 24x - 90y + 153 = 0$

21. $16y^2 - 25x^2 - 96y + 100x - 356 = 0$

22. $36x^2 - 49y^2 - 72x - 294y = 2169$

23. Graph the equation $25y^2 - 9x^2 - 100y - 72x - 269 = 0$. Label the center, foci and the equations of the asymptotes.

Write the equation of each hyperbola.

24.


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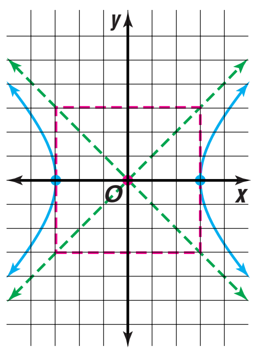

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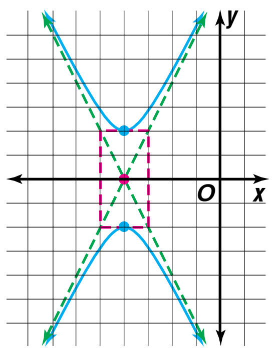

Graph each equation.

27. $xy = 49$

28. $xy = -36$

29. $4xy = -25$

30. $9xy = 16$

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Write an equation of the hyperbola that meets each set of conditions.

31. The center is at $(4, -2)$, $a = 2$, $b = 3$, and it has a vertical transverse axis.

32. The vertices are at $(0, 3)$ and $(0, -3)$, and a focus is at $(0, -9)$.

33. The length of the transverse axis is 6 units, and the foci are at $(5, 2)$ and $(-5, 2)$.

34. The length of the conjugate axis is 8 units, and the vertices are at $(-3, 9)$ and $(-3, -5)$.

35. The hyperbola is equilateral and has foci at $(8, 0)$ and $(-8, 0)$.

36. The center is at $(-3, 1)$, one focus is at $(2, 1)$, and the eccentricity is $\frac{5}{4}$.

37. A vertex is at $(4, 5)$, the center is at $(4, 2)$, and an equation of one asymptote is $4y + 4 = 3x$.

38. The equation of one asymptote is $3x - 11 = 2y$. The hyperbola has its center at $(3, -1)$ and a vertex at $(5, -1)$.

39. The hyperbola has foci at $(0, 8)$ and $(0, -8)$ and eccentricity $\frac{4}{3}$.

40. The hyperbola has eccentricity $\frac{6}{5}$ and foci at $(10, -3)$ and $(-2, -3)$.

41. The hyperbola is equilateral and has foci at $(9, 0)$ and $(-9, 0)$.

42. The slopes of the asymptotes are ± 2, and the foci are at $(1, 5)$ and $(1, -3)$.

Applications and Problem Solving

Real World Application

43. Chemistry According to Boyle's Law, the pressure P (in kilopascals) exerted by a gas varies inversely as the volume V (in cubic decimeters) of a gas if the temperature remains constant. That is, $PV = c$. Suppose the constant for oxygen at 25°C is 505.

a. Graph the function $PV = c$ for $c = 505$.

b. Determine the volume of oxygen if the pressure is 101 kilopascals.

c. Determine the volume of oxygen if the pressure is 50.5 kilopascals.

d. Study your results for parts **b** and **c**. If the pressure is halved, make a conjecture about the effect on the volume of gas.

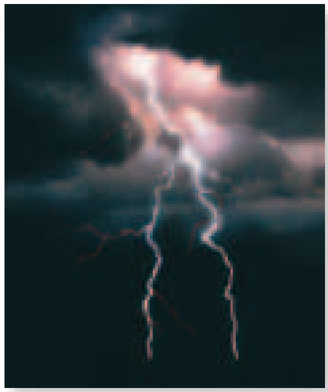
44. Critical Thinking Prove that the eccentricity of all equilateral hyperbolas is $\sqrt{2}$.

45. Nuclear Power A nuclear cooling tower is a *hyperboloid,* that is, a hyperbola rotated around its conjugate axis. Suppose the hyperbola used to generate the hyperboloid modeling the shape of the cooling tower has an eccentricity of $\frac{5}{3}$.

a. If the cooling tower is 150 feet wide at its narrowest point, determine an equation of the hyperbola used to generate the hyperboloid.

b. If the tower is 450 feet tall, the top is 100 feet above the center of the hyperbola, and the base is 350 feet below the center, what is the radius of the top and the base of the tower?

46. **Forestry** Two ranger stations located 4 miles apart observe a lightning strike. A ranger at station A reports hearing the sound of thunder 2 seconds prior to a ranger at station B. If sound travels at 1100 feet per second, determine the equation of the hyperbola on which the lightning strike was located. Place the two ranger stations on the x-axis with the midpoint between the two stations at the origin. The transverse axis is horizontal.



47. **Critical Thinking** A hyperbola has foci $F_1(-6, 0)$ and $F_1(6, 0)$. For any point $P(x, y)$ on the hyperbola, $|PF_1 - PF_2| = 10$. Write the equation of the hyperbola in standard form.

48. **Analytic Geometry** Two hyperbolas in which the transverse axis of one is the conjugate axis of the other are called *conjugate hyperbolas.* In equations of conjugate hyperbolas, the x^2 and y^2 terms are reversed. For example, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{16} = 1$ are equations of conjugate hyperbolas.
   a. Graph $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{16} = 1$ on the same coordinate plane.
   b. What is true of the asymptotes of conjugate hyperbolas?
   c. Write the equation of the conjugate hyperbola for $\frac{(x-3)^2}{16} - \frac{(y-2)^2}{25} = 1$.
   d. Graph the conjugate hyperbolas in part **c**.

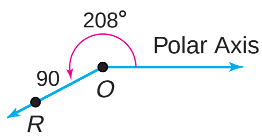
Mixed Review

49. Write the equation of the ellipse that has a semi-major axis length of 4 units and foci at (2, 3) and (2, −3). *(Lesson 10-3)*

50. Write $x^2 + y^2 - 4x + 14y - 28 = 0$ in standard form. Then graph the equation. *(Lesson 10-2)*

51. Show that the points with coordinates (−1, 3), (3, 6), (6, 2), and (2, −1) are the vertices of a square. *(Lesson 10-1)*

52. Name three different pairs of polar coordinates that represent point R. Assume $-360° \leq \theta \leq 360°$. *(Lesson 9-1)*



53. Find the inner product of vectors ⟨4, −1, 8⟩ and ⟨−5, 2, 2⟩. Are the vectors perpendicular? Explain. *(Lesson 8-4)*

54. Write the standard from of the equation of the line that has a normal 3 units long and makes an angle of 60° with the positive x-axis. *(Lesson 7-6)*

55. **Aviation** An airplane flying at an altitude of 9000 meters passes directly overhead. Fifteen seconds later, the angle of elevation to the plane is 60°. How fast is the airplane flying? *(Lesson 5-4)*

56. Approximate the real zeros of the function $f(x) = 4x^4 + 5x^3 - x^2 + 1$ to the nearest tenth. *(Lesson 4-5)*

57. **SAT/ACT Practice** If r and s are integers and $r + s = 0$, which of the following must be true?
   I. $r^3 > s^3$ II. $r^3 = s^3$ III. $r^4 = s^4$
   A I only **B** II only **C** III only
   D I and II only **E** I and III only

Extra Practice See p. A45.