6-6 Modeling Real-World Data with Sinusoidal Functions

OBJECTIVES

- Model real-world data using sine and cosine functions.
- Use sinusoidal functions to solve problems.

Real World Application

METEOROLOGY The table contains the times that the sun rises and sets on the fifteenth of every month in Brownsville, Texas.

Let $t = 1$ represent January 15.
Let $t = 2$ represent February 15.
Let $t = 3$ represent March 15.
⋮

Write a function that models the hours of daylight for Brownsville. Use your model to estimate the number of hours of daylight on September 30. *This problem will be solved in Example 1.*

Month	Sunrise A.M.	Sunset P.M.
January	7:19	6:00
February	7:05	6:23
March	6:40	6:39
April	6:07	6:53
May	5:44	7:09
June	5:38	7:23
July	5:48	7:24
August	6:03	7:06
September	6:16	6:34
October	6:29	6:03
November	6:48	5:41
December	7:09	5:41

Before you can determine the function for the daylight, you must first compute the amount of daylight for each day as a decimal value. Consider January 15. First, write each time in 24-hour time.

$$7{:}19 \text{ A.M.} = 7{:}19$$

$$6{:}00 \text{ P.M.} = 6{:}00 + 12 \text{ or } 18{:}00$$

Then change each time to a decimal rounded to the nearest hundredth.

$$7{:}19 = 7 + \frac{19}{60} \text{ or } 7.32$$

$$18{:}00 = 18 + \frac{0}{60} \text{ or } 18.00$$

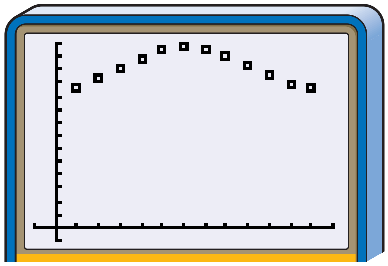
On January 15, there will be $18.00 - 7.32$ or 10.68 hours of daylight.

Similarly, the number of daylight hours can be determined for the fifteenth of each month.

Month	Jan.	Feb.	March	April	May	June
t	1	2	3	4	5	6
Hours of Daylight	10.68	11.30	11.98	12.77	13.42	13.75

Month	July	Aug.	Sept.	Oct.	Nov.	Dec.
t	7	8	9	10	11	12
Hours of Daylight	13.60	13.05	12.30	11.57	10.88	10.53

Since there are 12 months in a year, month 13 is the same as month 1, month 14 is the same as month 2, and so on. The function is periodic. Enter the data into a graphing calculator and graph the points. The graph resembles a type of sine curve. You can write a **sinusoidal function** to represent the data. A sinusoidal function can be any function of the form
$y = A \sin (k\theta + c) + h$ or
$y = A \cos (k\theta + c) + h$.



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Example 1 **METEOROLOGY** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Write a function that models the amount of daylight for Brownsville.

b. Use your model to estimate the number of hours of daylight on September 30.

a. The data can be modeled by a function of the form $y = A \sin (kt + c) + h$, where t is the time in months. First, find A, h, and k.

A: $A = \dfrac{13.75 - 10.53}{2}$ or 1.61 — *A is half the difference between the most daylight (13.75 h) and the least daylight (10.53 h).*

h: $h = \dfrac{13.75 + 10.53}{2}$ or 12.14 — *h is half the sum of the greatest value and least value.*

k: $\dfrac{2\pi}{k} = 12$ — *The period is 12.*

$k = \dfrac{\pi}{6}$

Substitute these values into the general form of the sinusoidal function.

$y = A \sin (kt + c) + h$

$y = 1.61 \sin \left(\dfrac{\pi}{6}t + c\right) + 12.14$ — *$A = 1.61$, $k = \dfrac{\pi}{6}$, $h = 12.14$*

To compute c, substitute one of the coordinate pairs into the function.

$$y = 1.61 \sin \left(\frac{\pi}{6}t + c\right) + 12.14$$

$$10.68 = 1.61 \sin \left(\frac{\pi}{6}(1) + c\right) + 12.14 \qquad \textit{Use } (t, y) = (1, 10.68).$$

$$-1.46 = 1.61 \sin \left(\frac{\pi}{6} + c\right) \qquad \textit{Add } -12.14 \textit{ to each side.}$$

$$-\frac{1.46}{1.61} = \sin \left(\frac{\pi}{6} + c\right) \qquad \textit{Divide each side by 1.61.}$$

$$\sin^{-1}\left(-\frac{1.46}{1.61}\right) = \frac{\pi}{6} + c \qquad \textit{Definition of inverse}$$

$$\sin^{-1}\left(-\frac{1.46}{1.61}\right) - \frac{\pi}{6} = c \qquad \textit{Add } -\frac{\pi}{6} \textit{ to each side.}$$

$$-1.659305545 \approx c \qquad \textit{Use a calculator.}$$

interNET CONNECTION

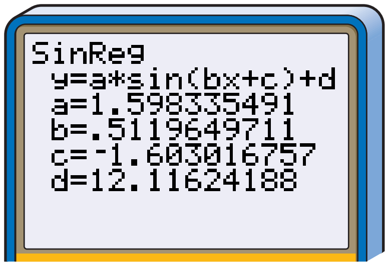
Research
For data about amount of daylight, average temperatures, or tides, visit **www.amc.glencoe.com**

The function $y = 1.61 \sin\left(\frac{\pi}{6}t - 1.66\right) + 12.14$ is one model for the daylight in Brownsville.

Graphing Calculator Tip

For keystroke instruction on how to find sine regression statistics, see page A25.

To check this answer, enter the data into a graphing calculator and calculate the **SinReg** statistics. Rounding to the nearest hundredth, $y = 1.60 \sin(0.51t - 1.60) + 12.12$. The models are similar. Either model could be used.



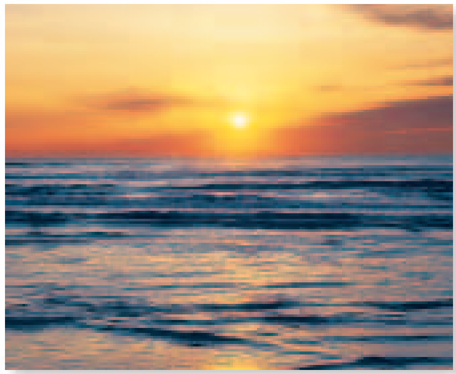
b. September 30 is half a month past September 15, so $t = 9.5$. Select a model and use a calculator to evaluate it for $t = 9.5$.

Model 1: Paper and Pencil

$y = 1.61 \sin\left(\frac{\pi}{6}t - 1.66\right) + 12.14$

$y = 1.61 \sin\left[\frac{\pi}{6}(9.5) - 1.66\right] + 12.14$

$y \approx 11.86349848$



Model 2: Graphing Calculator

$y = 1.60 \sin(0.51t - 1.60) + 12.12$

$y = 1.60 \sin[0.51(9.5) - 1.60] + 12.12$

$y \approx 11.95484295$

On September 30, Brownsville will have about 11.9 hours of daylight.

In general, any sinusoidal function can be written as a sine function or as a cosine function. The amplitude, the period, and the midline will remain the same. However, the phase shift will be different. To avoid a greater phase shift than necessary, you may wish to use a sine function if the function is about zero at $x = 0$ and a cosine function if the function is about the maximum or minimum at $x = 0$.

Example 2 Real World Application

HEALTH **An average seated adult breathes in and out every 4 seconds. The average minimum amount of air in the lungs is 0.08 liter, and the average maximum amount of air in the lungs is 0.82 liter. Suppose the lungs have a minimum amount of air at $t = 0$, where t is the time in seconds.**

a. Write a function that models the amount of air in the lungs.

b. Graph the function.

c. Determine the amount of air in the lungs at 5.5 seconds.

(continued on the next page)

a. Since the function has its minimum value at $t = 0$, use the cosine function. A cosine function with its minimum value at $t = 0$ has no phase shift and a negative value for A. Therefore, the general form of the model is $y = -A \cos kt + h$, where t is the time in seconds. Find A, k, and h.

***A*:** $A = \frac{0.82 - 0.08}{2}$ or 0.37 — *A is half the difference between the greatest value and the least value.*

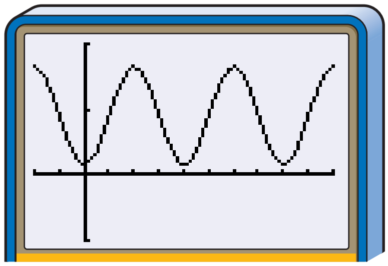
***h*:** $h = \frac{0.82 + 0.08}{2}$ or 0.45 — *h is half the sum of the greatest value and the least value.*

***k*:** $\frac{2\pi}{k} = 4$ — *The period is 4.*

$$k = \frac{\pi}{2}$$

Therefore, $y = -0.37 \cos \frac{\pi}{2} t + 0.45$ models the amount of air in the lungs of an average seated adult.

b. Use a graphing calculator to graph the function.



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c. Use this function to find the amount of air in the lungs at 5.5 seconds.

$$y = -0.37 \cos \frac{\pi}{2}t + 0.45$$

$$y = -0.37 \cos \left[\frac{\pi}{2}(5.5)\right] + 0.45$$

$$y \approx 0.711629509$$

The lungs have about 0.71 liter of air at 5.5 seconds.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Define** sinusoidal function in your own words.
2. **Compare and contrast** real-world data that can be modeled with a polynomial function and real-world data that can be modeled with a sinusoidal function.
3. **Give** three real-world examples that can be modeled with a sinusoidal function.

Guided Practice

4. **Boating** If the equilibrium point is $y = 0$, then $y = -5 \cos\left(\frac{\pi}{6}t\right)$ models a buoy bobbing up and down in the water.
   a. Describe the location of the buoy when $t = 0$.
   b. What is the maximum height of the buoy?
   c. Find the location of the buoy at $t = 7$.

5. **Health** A certain person's blood pressure oscillates between 140 and 80. If the heart beats once every second, write a sine function that models the person's blood pressure.

6. **Meteorology** The average monthly temperatures for the city of Seattle, Washington, are given below.

Jan.	Feb.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
41°	44°	47°	50°	56°	61°	65°	66°	61°	54°	46°	42°

   a. Find the amplitude of a sinusoidal function that models the monthly temperatures.
   b. Find the vertical shift of a sinusoidal function that models the monthly temperatures.
   c. What is the period of a sinusoidal function that models the monthly temperatures?
   d. Write a sinusoidal function that models the monthly temperatures, using $t = 1$ to represent January.
   e. According to your model, what is the average monthly temperature in February? How does this compare to the actual average?
   f. According to your model, what is the average monthly temperature in October? How does this compare to the actual average?

EXERCISES

Applications and Problem Solving

Real World Application

7. **Music** The initial behavior of the vibrations of the note E above middle C can be modeled by $y = 0.5 \sin 660\pi t$.
   a. What is the amplitude of this model?
   b. What is the period of this model?
   c. Find the frequency (cycles per second) for this note.

8. **Entertainment** A rodeo performer spins a lasso in a circle perpendicular to the ground. The height of the knot from the ground is modeled by $h = -3 \cos\left(\frac{5\pi}{3}t\right) + 3.5$, where t is the time measured in seconds.
   a. What is the highest point reached by the knot?
   b. What is the lowest point reached by the knot?
   c. What is the period of the model?
   d. According to the model, find the height of the knot after 25 seconds.

9. Biology In a certain region with hawks as predators and rodents as prey, the rodent population R varies according to the model $R = 1200 + 300 \sin\left(\frac{\pi}{2}t\right)$, and the hawk population H varies according to the model $H = 250 + 25 \sin\left(\frac{\pi}{2}t - \frac{\pi}{4}\right)$, with t measured in years since January 1, 1970.

a. What was the population of rodents on January 1, 1970?

b. What was the population of hawks on January 1, 1970?

c. What are the maximum populations of rodents and hawks? Do these maxima ever occur at the same time?

d. On what date was the first maximum population of rodents achieved?

e. What is the minimum population of hawks? On what date was the minimum population of hawks first achieved?

f. According to the models, what was the population of rodents and hawks on January 1 of the present year?

10. Waves A leaf floats on the water bobbing up and down. The distance between its highest and lowest point is 4 centimeters. It moves from its highest point down to its lowest point and back to its highest point every 10 seconds. Write a cosine function that models the movement of the leaf in relationship to the equilibrium point.

11. Tides Write a sine function which models the oscillation of tides in Savannah, Georgia, if the equilibrium point is 4.24 feet, the amplitude is 3.55 feet, the phase shift is -4.68 hours, and the period is 12.40 hours.

12. Meteorology The mean average temperature in Buffalo, New York, is 47.5°. The temperature fluctuates 23.5° above and below the mean temperature. If $t = 1$ represents January, the phase shift of the sine function is 4.

a. Write a model for the average monthly temperature in Buffalo.

b. According to your model, what is the average temperature in March?

c. According to your model, what is the average temperature in August?

13. **Meteorology** The average monthly temperatures for the city of Honolulu, Hawaii, are given below.

Jan.	Feb.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
73°	73°	74°	76°	78°	79°	81°	81°	81°	80°	77°	74°

a. Find the amplitude of a sinusoidal function that models the monthly temperatures.

b. Find the vertical shift of a sinusoidal function that models the monthly temperatures.

c. What is the period of a sinusoidal function that models the monthly temperatures?

d. Write a sinusoidal function that models the monthly temperatures, using $t = 1$ to represent January.

e. According to your model, what is the average temperature in August? How does this compare to the actual average?

f. According to your model, what is the average temperature in May? How does this compare to the actual average?

14. **Critical Thinking** Write a cosine function that is equivalent to $y = 3 \sin (x - \pi) + 5$.

15. **Tides** Burntcoat Head in Nova Scotia, Canada, is known for its extreme fluctuations in tides. One day in April, the first high tide rose to 13.25 feet at 4:30 A.M. The first low tide at 1.88 feet occurred at 10:51 A.M. The second high tide was recorded at 4:53 P.M.

a. Find the amplitude of a sinusoidal function that models the tides.

b. Find the vertical shift of a sinusoidal function that models the tides.

c. What is the period of a sinusoidal function that models the tides?

d. Write a sinusoidal function to model the tides, using t to represent the number of hours in decimals since midnight.

e. According to your model, determine the height of the water at 7:30 P.M.

16. **Meteorology** The table at the right contains the times that the sun rises and sets in the middle of each month in New York City, New York. Suppose the number 1 represents the middle of January, the number 2 represents the middle of February, and so on.

Month	Sunrise A.M.	Sunset P.M.
January	7:19	4:47
February	6:56	5:24
March	6:16	5:57
April	5.25	6:29
May	4:44	7:01
June	4:24	7:26
July	4:33	7:28
August	5:01	7:01
September	5:31	6:14
October	6:01	5:24
November	6:36	4:43
December	7:08	4:28

a. Find the amount of daylight hours for the middle of each month.

b. What is the amplitude of a sinusoidal function that models the daylight hours?

c. What is the vertical shift of a sinusoidal function that models the daylight hours?

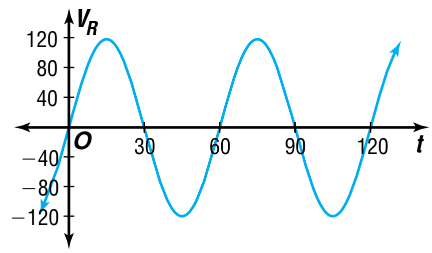
d. What is the period of a sinusoidal function that models the daylight hours?

e. Write a sinusoidal function that models the daylight hours.

17. **Critical Thinking** The average monthly temperature for Phoenix, Arizona can be modeled by $y = 70.5 + 19.5 \sin\left(\frac{\pi}{6}t + c\right)$. If the coldest temperature occurs in January ($t = 1$), find the value of c.

18. **Entertainment** Several years ago, an amusement park in Sandusky, Ohio, had a ride called the Rotor in which riders stood against the walls of a spinning cylinder. As the cylinder spun, the floor of the ride dropped out, and the riders were held against the wall by the force of friction. The cylinder of the Rotor had a radius of 3.5 meters and rotated counterclockwise at a rate of 14 revolutions per minute. Suppose the center of rotation of the Rotor was at the origin of a rectangular coordinate system.
   a. If the initial coordinates of the hinges on the door of the cylinder are $(0, -3.5)$, write a function that models the position of the door at t seconds.
   b. Find the coordinates of the hinges on the door at 4 seconds.

19. **Electricity** For an alternating current, the instantaneous voltage V_R is graphed at the right. Write an equation for the instantaneous voltage.



20. **Meteorology** Find the number of daylight hours for the middle of each month or the average monthly temperature for your community. Write a sinusoidal function to model this data.

Mixed Review

21. State the amplitude, period, phase shift, and vertical shift for $y = -3 \cos(2\theta + \pi) + 5$. Then graph the function. *(Lesson 6-5)*

22. Find the values of θ for which $\cos\theta = 1$ is true. *(Lesson 6-3)*

23. Change 800° to radians. *(Lesson 6-1)*

24. **Geometry** The sides of a parallelogram are 20 centimeters and 32 centimeters long. If the longer diagonal measures 40 centimeters, find the measures of the angles of the parallelogram. *(Lesson 5-8)*

25. Decompose $\frac{2m + 16}{m^2 - 16}$ into partial fractions. *(Lesson 4-6)*

26. Find the value of k so that the remainder of $(2x^3 + kx^2 - x - 6) \div (x + 2)$ is zero. *(Lesson 4-3)*

27. Determine the interval(s) for which the graph of $f(x) = 2|x + 1| - 5$ is increasing and the intervals for which the graph is decreasing. *(Lesson 3-5)*

28. **SAT/ACT Practice** If one half of the female students in a certain school eat in the cafeteria and one third of the male students eat there, what fractional part of the student body eats in the cafeteria?

   A $\frac{5}{12}$ B $\frac{2}{5}$ C $\frac{3}{4}$ D $\frac{5}{6}$

   E not enough information given

Extra Practice See p. A37.

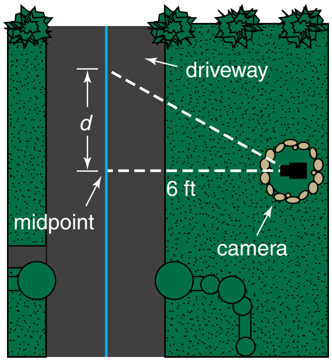
6-7

Graphing Other Trigonometric Functions

OBJECTIVES
- Graph tangent, cotangent, secant, and cosecant functions.
- Write equations of trigonometric functions.

Real World Application

SECURITY A security camera scans a long, straight driveway that serves as an entrance to an historic mansion. Suppose a line is drawn down the center of the driveway. The camera is located 6 feet to the right of the midpoint of the line. Let d represent the distance along the line from its midpoint. If t is time in seconds and the camera points at the midpoint at $t = 0$, then $d = 6 \tan\left(\frac{\pi}{30}t\right)$ models the point being scanned. In this model, the distance below the midpoint is a negative. Graph the equation for $-15 \leq t \leq 15$. Find the location the camera is scanning at 5 seconds. What happens when $t = 15$?



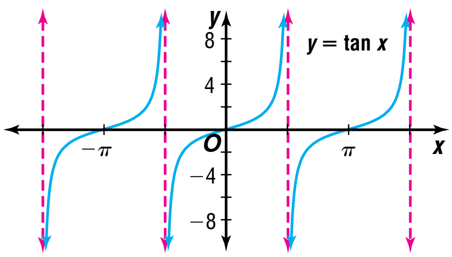
This problem will be solved in Example 4.

You have learned to graph variations of the sine and cosine functions. In this lesson, we will study the graphs of the tangent, cotangent, secant, and cosecant functions. Consider the tangent function. First evaluate $y = \tan x$ for multiples of $\frac{\pi}{4}$ in the interval $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$.

x	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$\tan x$	undefined	-1	0	1	undefined	-1	0	1	undefined	-1	0	1	undefined

Look Back
You can refer to Lesson 3-7 to review asymptotes.

To graph $y = \tan x$, draw the asymptotes and plot the coordinate pairs from the table. Then draw the curves.



Notice that the range values for the interval $-\frac{3\pi}{2} \leq x \leq -\frac{\pi}{2}$ repeat for the intervals $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$. So, the tangent function is a periodic function. Its period is π.

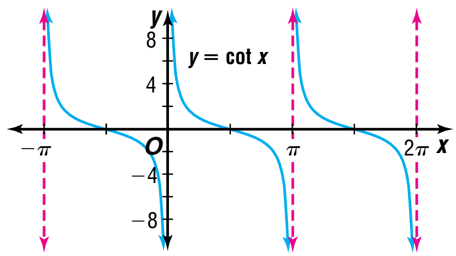
By studying the graph and its repeating pattern, you can determine the following properties of the graph of the tangent function.

Properties of the Graph $y = \tan x$

1. The period is π.
2. The domain is the set of real numbers except $\frac{\pi}{2}n$, where n is an odd integer.
3. The range is the set of real numbers.
4. The x-intercepts are located at πn, where n is an integer.
5. The y-intercept is 0.
6. The asymptotes are $x = \frac{\pi}{2}n$, where n is an odd integer.

Now consider the graph of $y = \cot x$ in the interval $-\pi \leq x \leq 3\pi$.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cot x$	undefined	1	0	-1	undefined	1	0	-1	undefined	1	0	-1	undefined



By studying the graph and its repeating pattern, you can determine the following properties of the graph of the cotangent function.

Properties of the Graph of $y = \cot x$

1. The period is π.
2. The domain is the set of real numbers except πn, where n is an integer.
3. The range is the set of real numbers.
4. The x-intercepts are located at $\frac{\pi}{2}n$, where n is an odd integer.
5. There is no y-intercept.
6. The asymptotes are $x = \pi n$, where n is an integer.

Example 1 **Find each value by referring to the graphs of the trigonometric functions.**

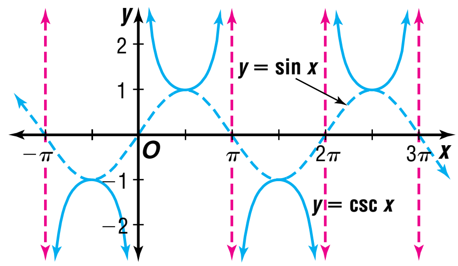
a. $\tan \frac{9\pi}{2}$

Since $\frac{9\pi}{2} = \frac{\pi}{2}(9)$, $\tan \frac{9\pi}{2}$ is undefined.

b. $\cot \frac{7\pi}{2}$

Since $\frac{7\pi}{2} = \frac{\pi}{2}(7)$ and 7 is an odd integer, $\cot \frac{7\pi}{2} = 0$.

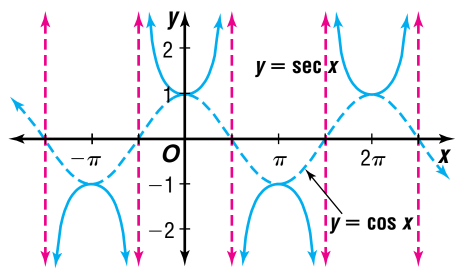
The sine and cosecant functions have a reciprocal relationship. To graph the cosecant, first graph the sine function and the asymptotes of the cosecant function. By studying the graph of the cosecant and its repeating pattern, you can determine the following properties of the graph of the cosecant function.



Properties of the Graph of $y = \csc x$

1. The period is 2π.
2. The domain is the set of real numbers except πn, where n is an integer.
3. The range is the set of real numbers greater than or equal to 1 or less than or equal to -1.
4. There are no x-intercepts.
5. There are no y-intercepts.
6. The asymptotes are $x = \pi n$, where n is an integer.
7. $y = 1$ when $x = \frac{\pi}{2} + 2\pi n$, where n is an integer.
8. $y = -1$ when $x = \frac{3\pi}{2} + 2\pi n$, where n is an integer.

The cosine and secant functions have a reciprocal relationship. To graph the secant, first graph the cosine function and the asymptotes of the secant function. By studying the graph and its repeating pattern, you can determine the following properties of the graph of the secant function.



Properties of the Graph of $y = \sec x$

1. The period is 2π.
2. The domain is the set of real numbers except $\frac{\pi}{2}n$, where n is an odd integer.
3. The range is the set of real numbers greater than or equal to 1 or less than or equal to -1.
4. There are no x-intercepts.
5. The y-intercept is 1.
6. The asymptotes are $x = \frac{\pi}{2}n$, where n is an odd integer.
7. $y = 1$ when $x = \pi n$, where n is an even integer.
8. $y = -1$ when $x = \pi n$, where n is an odd integer.

Example 2 **Find the values of θ for which each equation is true.**

a. $\csc \theta = 1$

From the pattern of the cosecant function, $\csc \theta = 1$ if $\theta = \frac{\pi}{2} + 2\pi n$, where n is an integer.

b. $\sec \theta = -1$

From the pattern of the secant function, $\sec \theta = -1$ if $\theta = \pi n$, where n is an odd integer.

The period of $y = \sin k\theta$ or $y = \cos k\theta$ is $\frac{2\pi}{k}$. Likewise, the period of $y = \csc k\theta$ or $y = \sec k\theta$ is $\frac{2\pi}{k}$. However, since the period of the tangent or cotangent function is π, the period of $y = \tan k\theta$ or $y = \cot k\theta$ is $\frac{\pi}{k}$. In each case, $k > 0$.

Period of Trigonometric Functions

The period of functions $y = \sin k\theta$, $y = \cos k\theta$, $y = \csc k\theta$, and $y = \sec k\theta$ is $\frac{2\pi}{k}$, where $k > 0$.

The period of functions $y = \tan k\theta$ and $y = \cot k\theta$ is $\frac{\pi}{k}$, where $k > 0$.

The phase shift and vertical shift work the same way for all trigonometric functions. For example, the phase shift of the function $y = \tan(k\theta + c) + h$ is $-\frac{c}{k}$, and its vertical shift is h.

Examples 3 **Graph** $y = \csc\left(\frac{\theta}{2} - \frac{\pi}{4}\right) + 2$.

The period is $\frac{2\pi}{\frac{1}{2}}$ or 4π. The phase shift is $-\frac{-\frac{\pi}{4}}{\frac{1}{2}}$ or $\frac{\pi}{2}$. The vertical shift is 2.

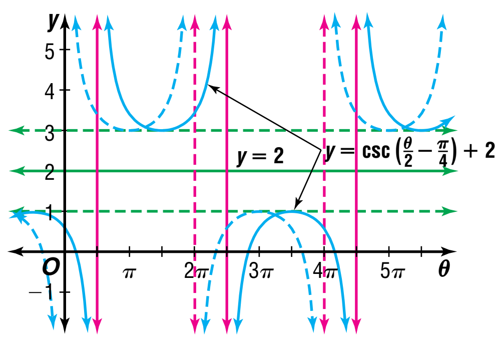
Use this information to graph the function.

Step 1 Draw the midline which is the graph of $y = 2$.

Step 2 Draw dashed lines parallel to the midline, which are 1 unit above and below the midline.

Step 3 Draw the cosecant curve with period of 4π.

Step 4 Shift the graph $\frac{\pi}{2}$ units to the right.



Real World Application

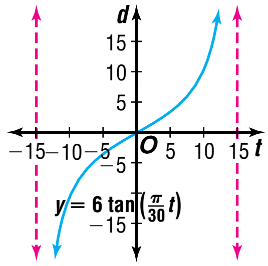
4 **SECURITY** **Refer to the application at the beginning of the lesson.**

a. Graph the equation $y = 6\tan\left(\frac{\pi}{30}t\right)$.

b. Find the location the camera is scanning after 5 seconds.

c. What happens when $t = 15$?

a. The period is $\frac{\pi}{\frac{\pi}{30}}$ or 30. There are no horizontal or vertical shifts. Draw the asymptotes at $t = -15$ and $t = 15$. Graph the equation.



b. Evaluate the equation at $t = 5$.

$d = 6\tan\left(\frac{\pi}{30}t\right)$

$d = 6\tan\left[\frac{\pi}{30}(5)\right]$ *$t = 5$*

$d \approx 3.464101615$ *Use a calculator.*

The camera is scanning a point that is about 3.5 feet above the center of the driveway.

c. At $\tan\left[\frac{\pi}{30}(15)\right]$ or $\tan\frac{\pi}{2}$, the function is undefined. Therefore, the camera will not scan any part of the driveway when $t = 15$. It will be pointed in a direction that is parallel with the driveway.

You can write an equation of a trigonometric function if you are given the period, phase shift, and vertical translation.

Example **5** **Write an equation for a secant function with period π, phase shift $\frac{\pi}{3}$, and vertical shift -3.**

The form of the equation will be $y = \sec(k\theta + c) + h$. Find the values of k, c, and h.

***k*:** $\frac{2\pi}{k} = \pi$ *The period is π.*

$k = 2$

***c*:** $-\frac{c}{k} = \frac{\pi}{3}$ *The phase shift is $\frac{\pi}{3}$.*

$-\frac{c}{2} = \frac{\pi}{3}$ *$k = 2$*

$c = -\frac{2\pi}{3}$

***h*:** $h = -3$

Substitute these values into the general equation. The equation is

$y = \sec\left(2\theta - \frac{2\pi}{3}\right) - 3.$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Name** three values of θ that would result in $y = \cot \theta$ being undefined.
2. **Compare** the asymptotes and periods of $y = \tan \theta$ and $y = \sec \theta$.
3. **Describe** two different phase shifts of the secant function that would make it appear to be the cosecant function.

Guided Practice

Find each value by referring to the graphs of the trigonometric functions.

4. $\tan 4\pi$
5. $\csc\left(-\frac{7\pi}{2}\right)$

Find the values of θ for which each equation is true.

6. $\sec \theta = -1$
7. $\cot \theta = 1$

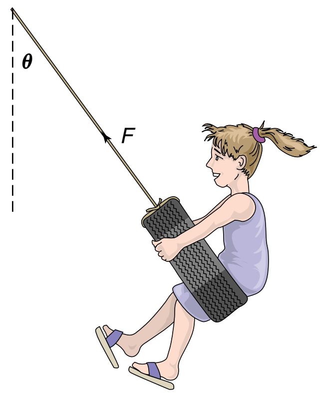
Graph each function.

8. $y = \tan\left(\theta + \frac{\pi}{4}\right)$
9. $y = \sec(2\theta + \pi) - 1$

Write an equation for the given function given the period, phase shift, and vertical shift.

10. cosecant function, period $= 3\pi$, phase shift $= \frac{\pi}{3}$, vertical shift $= -4$
11. cotangent function, period $= 2\pi$, phase shift $= -\frac{\pi}{4}$, vertical shift $= 0$

12. **Physics** A child is swinging on a tire swing. The tension on the rope is equal to the downward force on the end of the rope times $\sec \theta$, where θ is the angle formed by a vertical line and the rope.
    a. The downward force in newtons equals the mass of the child and the swing in kilograms times the acceleration due to gravity (9.8 meters per second squared). If the mass of the child and the tire is 73 kilograms, find the downward force.
    b. Write an equation that represents the tension on the rope as the child swings back and forth.
    c. Graph the equation for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
    d. What is the least amount of tension on the rope?
    e. What happens to the tension on the rope as the child swings higher and higher?



EXERCISES

Practice

Find each value by referring to the graphs of the trigonometric functions.

13. $\cot\left(\frac{5\pi}{2}\right)$
14. $\tan(-8\pi)$
15. $\sec\left(\frac{9\pi}{2}\right)$
16. $\csc\left(-\frac{5\pi}{2}\right)$
17. $\sec 7\pi$
18. $\cot(-5\pi)$

19. What is the value of csc (-6π)?

20. Find the value of tan (10π).

Find the values of θ for which each equation is true.

21. $\tan \theta = 0$

22. $\sec \theta = 1$

23. $\csc \theta = -1$

24. $\tan \theta = 1$

25. $\tan \theta = -1$

26. $\cot \theta = -1$

27. What are the values of θ for which sec θ is undefined?

28. Find the values of θ for which cot θ is undefined.

Graph each function.

29. $y = \cot\left(\theta - \frac{\pi}{2}\right)$

30. $y = \sec \frac{\theta}{3}$

31. $y = \csc \theta + 5$

32. $y = \tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right) + 1$

33. $y = \csc (2\theta + \pi) - 3$

34. $y = \sec\left(\frac{\theta}{3} + \frac{\pi}{6}\right) - 2$

35. Graph $y = \cos \theta$ and $y = \sec \theta$. In the interval of -2π and 2π, what are the values of θ where the two graphs are tangent to each other?

Write an equation for the given function given the period, phase shift, and vertical shift.

36. tangent function, period $= 2\pi$, phase shift $= 0$, vertical shift $= -6$

37. cotangent function, period $= \frac{\pi}{2}$, phase shift $= \frac{\pi}{8}$, vertical shift $= 7$

38. secant function, period $= \pi$, phase shift $= -\frac{\pi}{4}$, vertical shift $= -10$

39. cosecant function, period $= 3\pi$, phase shift $= \pi$, vertical shift $= -1$

40. cotangent function, period $= 5\pi$, phase shift $= -\pi$, vertical shift $= 12$

41. cosecant function, period $= \frac{\pi}{3}$, phase shift $= -\frac{\pi}{2}$, vertical shift $= -5$

42. Write a secant function with a period of 3π, a phase shift of π units to the left, and a vertical shift of 8 units downward.

43. Write a tangent function with a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift of 7 units upward.

Applications and Problem Solving

44. **Security** A security camera is scanning a long straight fence along one side of a military base. The camera is located 10 feet from the center of the fence. If d represents the distance along the fence from the center and t is time in seconds, then $d = 10 \tan \frac{\pi}{40} t$ models the point being scanned.

a. Graph the equation for $-20 \le t \le 20$.

b. Find the location the camera is scanning at 3 seconds.

c. Find the location the camera is scanning at 15 seconds.

45. **Critical Thinking** Graph $y = \csc \theta$, $y = 3 \csc \theta$, and $y = -3 \csc \theta$. Compare and contrast the graphs.

46. Physics A wire is used to hang a painting from a nail on a wall as shown at the right. The tension on each half of the wire is equal to half the downward force times $\sec \frac{\theta}{2}$.

a. The downward force in newtons equals the mass of the painting in kilograms times 9.8. If the mass of the painting is 7 kilograms, find the downward force.

b. Write an equation that represents the tension on each half of the wire.

c. Graph the equation for $0 \le \theta \le \pi$.

d. What is the least amount of tension on each side of the wire?

e. As the measure of θ becomes greater, what happens to the tension on each side of the wire?

47. Electronics The current I measured in amperes that is flowing through an alternating current at any time t in seconds is modeled by $I = 220 \sin\left(60\pi t - \frac{\pi}{6}\right)$.

a. What is the amplitude of the current?

b. What is the period of the current?

c. What is the phase shift of this sine function?

d. Find the current when $t = 60$.

48. Critical Thinking Write a tangent function that has the same graph as $y = \cot \theta$.

Mixed Review

49. Tides In Daytona Beach, Florida, the first high tide was 3.99 feet at 12:03 A.M. The first low tide of 0.55 foot occurred at 6:24 A.M. The second high tide occurred at 12:19 P.M. *(Lesson 6-6)*

a. Find the amplitude of a sinusoidal function that models the tides.

b. Find the vertical shift of the sinusoidal function that models the tides.

c. What is the period of the sinusoidal function that models the tides?

d. Write a sinusoidal function to model the tides, using t to represent the number of hours in decimals since midnight.

e. According to your model, determine the height of the water at noon.

50. Graph $y = 2 \cos \frac{\theta}{2}$. *(Lesson 6-4)*

51. If a central angle of a circle with radius 18 centimeters measures $\frac{\pi}{3}$, find the length (in terms of π) of its intercepted arc. *(Lesson 6-1)*

52. Solve $\triangle ABC$ if $A = 62°31'$, $B = 75°18'$, and $a = 57.3$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-6)*

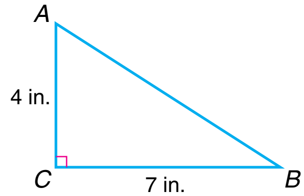
53. **Entertainment** A utility pole is braced by a cable attached to the top of the pole and anchored in a concrete block at the ground level 4 meters from the base of the pole. The angle between the cable and the ground is 73°. *(Lesson 5-4)*

a. Draw a diagram of the problem.

b. If the pole is perpendicular with the ground, what is the height of the pole?

c. Find the length of the cable.

54. Find the values of the sine, cosine, and tangent for $\angle A$. *(Lesson 5-2)*



55. Solve $\frac{x^2 - 4}{x^2 - 3x - 10} \leq 0$. *(Lesson 4-6)*

56. If r varies directly as t and $t = 6$ when $r = 0.5$, find r when $t = 10$. *(Lesson 3-8)*

57. Solve the system of inequalities by graphing. *(Lesson 2-6)*
$3x + 2y < 8$
$y < 2x + 1$
$-2y < -x + 4$

58. **Nutrition** The fat grams and Calories in various frozen pizzas are listed below. Use a graphing calculator to find the equation of the regression line and the Pearson product-moment correlation value. *(Lesson 1-6)*

Pizza	Fat (grams)	Calories
Cheese Pizza	14	270
Party Pizza	17	340
Pepperoni French Bread Pizza	22	430
Hamburger French Bread Pizza	19	410
Deluxe French Bread Pizza	20	420
Pepperoni Pizza	19	360
Sausage Pizza	18	360
Sausage and Pepperoni Pizza	18	340
Spicy Chicken Pizza	16	360
Supreme Pizza	18	308
Vegetable Pizza	13	300
Pizza Roll-Ups	13	250

59. **SAT/ACT Practice** The distance from City A to City B is 150 miles. From City A to City C is 90 miles. Which of the following is necessarily true?

A The distance from B to C is 60 miles.
B Six times the distance from A to B equals 10 times the distance from A to C.
C The distance from B to C is 240 miles.
D The distance from A to B exceeds by 30 miles twice the distance from A to C.
E Three times the distance from A to C exceeds by 30 miles twice the distance from A to B.

Extra Practice See p. A37.

6-7B Sound Beats

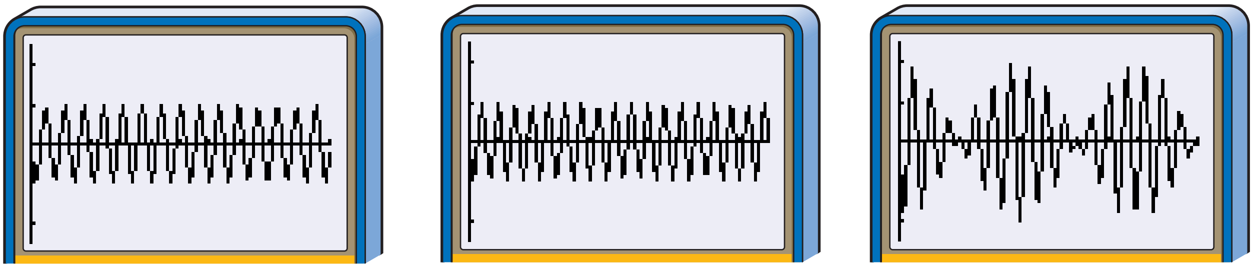
An Extension of Lesson 6-7

OBJECTIVE

- Use a graphing calculator to model beat effects produced by waves of almost equal frequencies.

The frequency of a wave is defined as the reciprocal of the period of the wave. If you listen to two steady sounds that have almost the same frequencies, you can detect an effect known as *beat.* Used in this sense, the word refers to a regular variation in sound intensity. This meaning is very different from another common meaning of the word, which you use when you are speaking about the rhythm of music for dancing.

A beat effect can be modeled mathematically by combination of two sine waves. The loudness of an actual combination of two steady sound waves of almost equal frequency depends on the amplitudes of the component sound waves. The first two graphs below picture two sine waves of almost equal frequencies. The amplitudes are equal, and the graphs, on first inspection, look almost the same. However, when the functions shown by the graphs are added, the resulting third graph is not what you would get by stretching either of the original graphs by a factor of 2, but is instead something quite different.



TRY THESE

1. Graph $f(x) = \sin(5\pi x) + \sin(4.79\pi x)$ using a window **[0, 10π] scl:π** by **[−2.5, 2.5] scl:1**. Which of the graphs shown above does the graph resemble?
2. Change the window settings for the independent variable to have **Xmax** $= 200\pi$. How does the appearance of the graph change?
3. For the graph in Exercise 2, use value on the **CALC** menu to find the value of $f(x)$ when $x = 187.158$.
4. Does your graph of Exercise 2 show negative values of y when x is close to 187.158?
5. Use value on the **CALC** menu to find $f(191.5)$. Does your result have any bearing on your answer for Exercise 4? Explain.

WHAT DO YOU THINK?

6. What aspect of the calculator explains your observations in Exercises 3-5?
7. Write two sine functions with almost equal frequencies. Graph the sum of the two functions. Discuss any interesting features of the graph.
8. Do functions that model beat effects appear to be periodic functions? Do your graphs prove that your answer is correct?

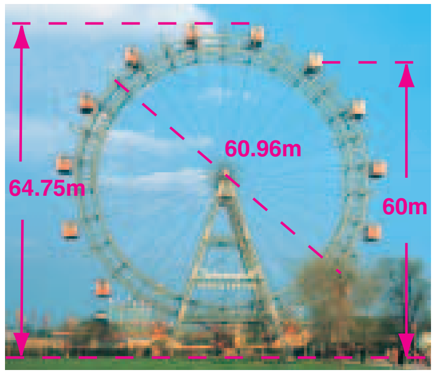
6-8

Trigonometric Inverses and Their Graphs

OBJECTIVES
- Graph inverse trigonometric functions.
- Find principal values of inverse trigonometric functions.

Real World Application

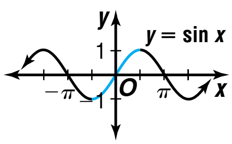
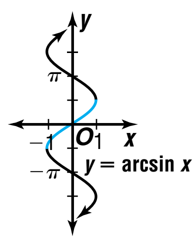
ENTERTAINMENT Since the giant Ferris wheel in Vienna, Austria, was completed in 1897, it has been a major attraction for local residents and tourists. The giant Ferris wheel has a height of 64.75 meters and a diameter of 60.96 meters. It makes a revolution every 4.25 minutes. On her summer vacation in Vienna, Carla starts timing her ride at the midline point at exactly 11:35 A.M. as she is on her way up. When Carla reaches an altitude of 60 meters, she will have a view of the Vienna Opera House. When will she have this view for the first time? *This problem will be solved in Example 4.*



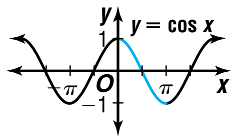
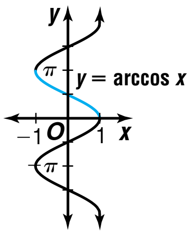
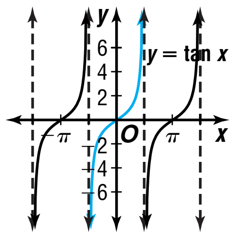
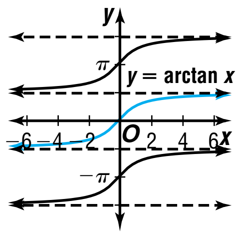
Look Back
You can refer to Lesson 5-5 to review the inverses of trigonometric functions.

Recall that the inverse of a function may be found by interchanging the coordinates of the ordered pairs of the function. In other words, the domain of the function becomes the range of its inverse, and the range of the function becomes the domain of its inverse. For example, the inverse of $y = 2x + 5$ is $x = 2y + 5$ or $y = \frac{x - 5}{2}$. Also remember that the inverse of a function may *not* be a function.

Consider the sine function and its inverse.

Relation	Ordered Pairs	Graph	Domain	Range
$y = \sin x$	$(x, \sin x)$		all real numbers	$-1 \le y \le 1$
$y = \arcsin x$	$(\sin x, x)$		$-1 \le x \le 1$	all real numbers

Notice the similarity of the graph of the inverse of the sine function to the graph of $y = \sin x$ with the axes interchanged. This is also true for the other trigonometric functions and their inverses.

Relation	Ordered Pairs	Graph	Domain	Range
$y = \cos x$	$(x, \cos x)$		all real numbers	$-1 \leq y \leq 1$
$y = \arccos x$	$(\cos x, x)$		$-1 \leq x \leq 1$	all real numbers
$y = \tan x$	$(x, \tan x)$		all real numbers except $\frac{\pi}{2}n$, where n is an odd integer	all real numbers
$y = \arctan x$	$(\tan x, x)$		all real numbers	all real numbers except $\frac{\pi}{2}n$, where n is an odd integer

Notice that none of the inverses of the trigonometric functions are functions.

Capital letters are used to distinguish the function with restricted domains from the usual trigonometric functions.

Consider only a part of the domain of the sine function, namely $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The range then contains all of the possible values from -1 to 1. It is possible to define a new function, called Sine, whose inverse is a function.

$$y = \text{Sin } x \text{ if and only if } y = \sin x \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

The values in the domain of Sine are called **principal values.** Other new functions can be defined as follows.

$$y = \text{Cos } x \text{ if and only if } y = \cos x \text{ and } 0 \leq x \leq \pi.$$

$$y = \text{Tan } x \text{ if and only if } y = \tan x \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The graphs of $y = \text{Sin } x$, $y = \text{Cos } x$, and $y = \text{Tan } x$ are the blue portions of the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$, respectively, shown on pages 405–406.

Note the capital "A" in the name of each inverse function.

The inverses of the Sine, Cosine, and Tangent functions are called Arcsine, Arccosine, and Arctangent, respectively. The graphs of Arcsine, Arccosine, and Arctangent are also designated in blue on pages 405–406. They are defined as follows.

Arcsine Function — Given $y = \text{Sin } x$, the inverse Sine function is defined by the equation $y = \text{Sin}^{-1} x$ or $y = \text{Arcsin } x$.

Arccosine Function — Given $y = \text{Cos } x$, the inverse Cosine function is defined by the equation $y = \text{Cos}^{-1} x$ or $y = \text{Arccos } x$.

Arctangent Function — Given $y = \text{Tan } x$, the inverse Tangent function is defined by the equation $y = \text{Tan}^{-1} x$ or $y = \text{Arctan } x$.

The domain and range of these functions are summarized below.

Function	Domain	Range
$y = \text{Sin } x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$-1 \le y \le 1$
$y = \text{Arcsin } x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \text{Cos } x$	$0 \le x \le \pi$	$-1 \le y \le 1$
$y = \text{Arccos } x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \text{Tan } x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	all real numbers
$y = \text{Arctan}$	all real numbers	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Example 1 **Write the equation for the inverse of $y = \text{Arctan } 2x$. Then graph the function and its inverse.**

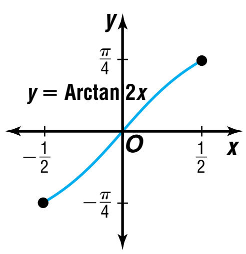
$y = \text{Arctan } 2x$

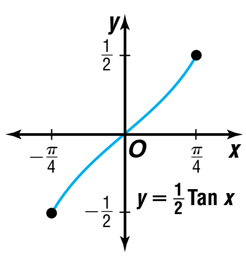
$x = \text{Arctan } 2y$ *Exchange x and y.*

$\text{Tan } x = 2y$ *Definition of Arctan function*

$\frac{1}{2}\text{Tan } x = y$ *Divide each side by 2.*

Now graph the functions.





Note that the graphs are reflections of each other over the graph of $y = x$.

You can use what you know about trigonometric functions and their inverses to evaluate expressions.

Examples **2** **Find each value.**

a. $\textbf{Arcsin}\left(-\frac{\sqrt{2}}{2}\right)$

Let $\theta = \text{Arcsin}\left(-\frac{\sqrt{2}}{2}\right)$. *Think: Arcsin* $\left(-\frac{\sqrt{2}}{2}\right)$ *means that angle whose sin is* $-\frac{\sqrt{2}}{2}$.

$\text{Sin}\ \theta = -\frac{\sqrt{2}}{2}$ *Definition of Arcsin function*

$\theta = -\frac{\pi}{4}$ *Why is* θ *not* $-\frac{3\pi}{4}$*?*

b. $\textbf{Sin}^{-1}\left(\textbf{cos}\ \frac{\pi}{2}\right)$

If $y = \cos \frac{\pi}{2}$, then $y = 0$.

$\text{Sin}^{-1}\left(\cos \frac{\pi}{2}\right) = \text{Sin}^{-1}\ 0$ *Replace* $\cos \frac{\pi}{2}$ *with 0.*

$= 0$

c. $\textbf{sin}\ (\textbf{Tan}^{-1}\ 1 - \textbf{Sin}^{-1}\ 1)$

Let $\alpha = \text{Tan}^{-1}\ 1$ and $\beta = \text{Sin}^{-1}\ 1$.

$\text{Tan}\ \alpha = 1$ $\quad$ $\text{Sin}\ \beta = 1$

$\alpha = \frac{\pi}{4}$ $\quad$ $\beta = \frac{\pi}{2}$

$$\begin{aligned}\sin\ (\text{Tan}^{-1}\ 1 - \text{Sin}^{-1}\ 1) &= \sin\ (\alpha - \beta)\\ &= \sin\left(\frac{\pi}{4} - \frac{\pi}{2}\right) && \alpha = \frac{\pi}{4}, \beta = \frac{\pi}{2}\\ &= \sin\left(-\frac{\pi}{4}\right)\\ &= -\frac{\sqrt{2}}{2}\end{aligned}$$

d. $\textbf{cos}\left[\textbf{Cos}^{-1}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\pi}{2}\right]$

Let $\theta = \text{Cos}^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

$\text{Cos}\ \theta = -\frac{\sqrt{2}}{2}$ *Definition of Arccosine function*

$\theta = \frac{3\pi}{4}$

$$\begin{aligned}\cos\left[\text{Cos}^{-1}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\pi}{2}\right] &= \cos\left(\theta - \frac{\pi}{2}\right)\\ &= \cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) && \theta = \frac{3\pi}{4}\\ &= \cos \frac{\pi}{4}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

3 **Determine if $\text{Tan}^{-1}(\tan x) = x$ is *true* or *false* for all values of x. If false, give a counterexample.**

Try several values of x to see if we can find a counterexample.

When $x = \pi$, $\text{Tan}^{-1}(\tan x) \neq x$. So $\text{Tan}^{-1}(\tan x) = x$ is not true for all values of x.

x	$\tan x$	$\text{Tan}^{-1}(\tan x)$
0	0	0
$\frac{\pi}{4}$	1	$\frac{\pi}{4}$
π	0	0

You can use a calculator to find inverse trigonometric functions. The calculator will always give the least, or principal, value of the inverse trigonometric function.

Example Real World Application

4 **ENTERTAINMENT** **Refer to the application at the beginning of the lesson. When will Carla reach an altitude of 60 meters for the first time?**

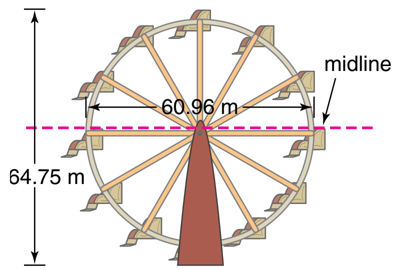
First write an equation to model the height of a seat at any time t. Since the seat is at the midline point at $t = 0$, use the sine function $y = A \sin(kt + c) + h$. Find the values of A, k, c, and h.

***A*:** The value of A is the radius of the Ferris wheel.

$A = \frac{1}{2}(60.96)$ or 30.48 *The diameter is 60.96 meters.*

***k*:** $\frac{2\pi}{k} = 4.25$ *The period is 4.25 minutes.*

$k = \frac{2\pi}{4.25}$



***c*:** Since the seat is at the equilibrium point at $t = 0$, there is no phase shift and $c = 0$.

***h*:** The bottom of the Ferris wheel is $64.75 - 60.96$ or 3.79 meters above the ground. So, the value of h is $30.48 + 3.79$ or 34.27.

Substitute these values into the general equation. The equation is $y = 30.48 \sin\left(\frac{2\pi}{4.25}t\right) + 34.27$. Now, solve the equation for $y = 60$.

$$60 = 30.48 \sin\left(\frac{2\pi}{4.25}t\right) + 34.27 \quad \textit{Replace y with 60.}$$

$$25.73 = 30.48 \sin\left(\frac{2\pi}{4.25}t\right) \quad \textit{Subtract 34.27 from each side.}$$

$$\frac{25.73}{30.48} = \sin\left(\frac{2\pi}{4.25}t\right) \quad \textit{Divide each side by 30.48.}$$

$$\sin^{-1}\left(\frac{25.73}{30.48}\right) = \frac{2\pi}{4.25}t \quad \textit{Definition of } \sin^{-1}$$

$$\frac{4.25}{2\pi}\sin^{-1}\left(\frac{25.73}{30.48}\right) = t \quad \textit{Multiply each side by } \frac{4.25}{2\pi}.$$

$$0.6797882017 = t \quad \textit{Use a calculator.}$$

Carla will reach an altitude of 60 meters about 0.68 minutes after 11:35 or 11:35:41.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare** $y = \sin^{-1} x$, $y = (\sin x)^{-1}$, and $y = \sin(x^{-1})$.
2. **Explain** why $y = \cos^{-1} x$ is not a function.
3. **Compare and contrast** the domain and range of $y = \text{Sin } x$ and $y = \sin x$.
4. **Write** a sentence explaining how to tell if the domain of a trigonometric function is restricted.
5. **You Decide** Jake says that the period of the cosine function is 2π. Therefore, he concludes that the principal values of the domain are between 0 and 2π, inclusive. Akikta disagrees. Who is correct? Explain.

Guided Practice

Write the equation for the inverse of each function. Then graph the function and its inverse.

6. $y = \text{Arcsin } x$
7. $y = \text{Cos}\left(x + \frac{\pi}{2}\right)$

Find each value.

8. Arctan 1
9. $\cos(\text{Tan}^{-1} 1)$
10. $\cos\left[\text{Cos}^{-1}\left(\frac{\sqrt{2}}{2}\right) - \frac{\pi}{2}\right]$

Determine if each of the following is *true* or *false*. If false, give a counterexample.

11. $\sin(\text{Sin}^{-1} x) = x$ for $-1 \leq x \leq 1$
12. $\text{Cos}^{-1}(-x) = -\text{Cos}^{-1} x$ for $-1 \leq x \leq 1$

13. **Geography** Earth has been charted with vertical and horizontal lines so that points can be named with coordinates. The horizontal lines are called latitude lines. The equator is latitude line 0. Parallel lines are numbered up to $\frac{\pi}{2}$ to the north and to the south. If we assume Earth is spherical, the length of any parallel of latitude is equal to the circumference of a great circle of Earth times the cosine of the latitude angle.
    a. The radius of Earth is about 6400 kilometers. Find the circumference of a great circle.
    b. Write an equation for the circumference of any latitude circle with angle θ.
    c. Which latitude circle has a circumference of about 3593 kilometers?
    d. What is the circumference of the equator?

EXERCISES

Practice

Write the equation for the inverse of each function. Then graph the function and its inverse.

14. $y = \arccos x$
15. $y = \text{Sin } x$
16. $y = \arctan x$
17. $y = \text{Arccos } 2x$
18. $y = \frac{\pi}{2} + \text{Arcsin } x$
19. $y = \tan \frac{x}{2}$
20. Is $y = \text{Tan}^{-1}\left(x + \frac{\pi}{2}\right)$ the inverse of $y = \text{Tan}\left(x - \frac{\pi}{2}\right)$? Explain.

21. The principal values of the domain of the cotangent function are $0 \leq x \leq \pi$. Graph $y = \text{Cot } x$ and its inverse.

Find each value.

22. $\text{Sin}^{-1}\, 0$

23. Arccos 0

24. $\text{Tan}^{-1} \frac{\sqrt{3}}{3}$

25. $\text{Sin}^{-1} \left(\tan \frac{\pi}{4}\right)$

26. $\sin \left(2 \text{ Cos}^{-1} \frac{\sqrt{2}}{2}\right)$

27. $\cos\, (\text{Tan}^{-1} \sqrt{3})$

28. $\cos\, (\text{Tan}^{-1}\, 1 - \text{Sin}^{-1}\, 1)$

29. $\cos \left(\text{Cos}^{-1}\, 0 + \text{Sin}^{-1} \frac{1}{2}\right)$

30. $\sin \left(\text{Sin}^{-1}\, 1 - \text{Cos}^{-1} \frac{1}{2}\right)$

31. Is it possible to evaluate $\cos \left[\text{Cos}^{-1} \left(-\frac{1}{2}\right) - \text{Sin}^{-1}\, 2\right]$? Explain.

Determine if each of the following is *true* or *false.* If false, give a counterexample.

32. $\text{Cos}^{-1}\, (\cos x) = x$ for all values of x

33. $\tan\, (\text{Tan}^{-1}\, x) = x$ for all values of x

34. $\text{Arccos } x = \text{Arccos } (-x)$ for $-1 \leq x \leq 1$

35. $\text{Sin}^{-1}\, x = -\text{Sin}^{-1}\, (-x)$ for $-1 \leq x \leq 1$

36. $\text{Sin}^{-1}\, x + \text{Cos}^{-1}\, x = \frac{\pi}{2}$ for $-1 \leq x \leq 1$

37. $\text{Cos}^{-1}\, x = \frac{1}{\text{Cos } x}$ for all values of x

38. Sketch the graph of $y = \tan\, (\text{Tan}^{-1}\, x)$.

Applications and Problem Solving

39. Meteorology The equation $y = 54.5 + 23.5 \sin \left(\frac{\pi}{6}t - \frac{2\pi}{3}\right)$ models the average monthly temperatures of Springfield, Missouri. In this equation, t denotes the number of months with January represented by 1. During which two months is the average temperature 54.5°?

40. Physics The average power P of an electrical circuit with alternating current is determined by the equation $P = VI \text{ Cos } \theta$, where V is the voltage, I is the current, and θ is the measure of the phase angle. A circuit has a voltage of 122 volts and a current of 0.62 amperes. If the circuit produces an average of 7.3 watts of power, find the measure of the phase angle.

41. Critical Thinking Consider the graphs $y = \arcsin x$ and $y = \arccos x$. Name the y coordinates of the points of intersection of the two graphs.

42. Optics Malus' Law describes the amount of light transmitted through two polarizing filters. If the axes of the two filters are at an angle of θ radians, the intensity I of the light transmitted through the filters is determined by the equation $I = I_0 \cos^2 \theta$, where I_0 is the intensity of the light that shines on the filters. At what angle should the axes be held so that one-eighth of the transmitted light passes through the filters?

43. **Tides** One day in March in Hilton Head, South Carolina, the first high tide occurred at 6:18 A.M. The high tide was 7.05 feet, and the low tide was -0.30 feet. The period for the oscillation of the tides is 12 hours and 24 minutes.

a. Determine what time the next high tide will occur.

b. Write the period of the oscillation as a decimal.

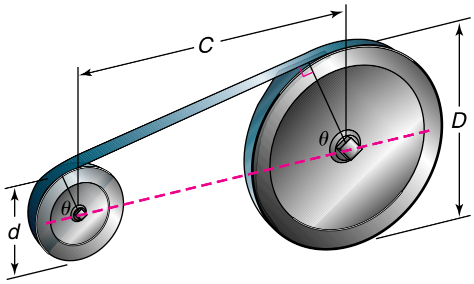
c. What is the amplitude of the sinusoidal function that models the tide?

d. If $t = 0$ represents midnight, write a sinusoidal function that models the tide.

e. At what time will the tides be at 6 feet for the first time that day?

44. **Critical Thinking** Sketch the graph of $y = \sin(\text{Tan}^{-1} x)$.

45. **Engineering** The length L of the belt around two pulleys can be determined by the equation $L = \pi D + (d - D)\,\theta + 2C \sin \theta$, where D is the diameter of the larger pulley, d is the diameter of the smaller pulley, and C is the distance between the centers of the two pulleys. In this equation, θ is measured in radians and equals $\cos^{-1} \dfrac{D - d}{2C}$.



a. If $D = 6$ inches, $d = 4$ inches, and $C = 10$ inches, find θ.

b. What is the length of the belt needed to go around the two pulleys?

Mixed Review

θ

46. What are the values of θ for which $\csc \theta$ is undefined? *(Lesson 6-7)*

47. Write an equation of a sine function with amplitude 5, period 3π, phase shift $-\pi$, and vertical shift -8. *(Lesson 6-5)*

48. Graph $y = \cos x$ for $-11\pi \leq x \leq -9\pi$. *(Lesson 6-3)*

49. **Geometry** Each side of a rhombus is 30 units long. One diagonal makes a $25°$ angle with a side. What is the length of each diagonal to the nearest tenth of a unit? *(Lesson 5-6)*

50. Find the measure of the reference angle for an angle of $210°$. *(Lesson 5-1)*

51. List the possible rational zeros of $f(x) = 2x^3 - 9x^2 - 18x + 6$. *(Lesson 4-4)*

52. Graph $y = \dfrac{1}{x-2} + 3$. Determine the interval(s) for which the function is increasing and the interval(s) for which the function is decreasing. *(Lesson 3-5)*

53. Find $[f \circ g](x)$ and $[g \circ f](x)$ if $f(x) = x^3 - 1$ and $g(x) = 3x$. *(Lesson 1-2)*

54. **SAT/ACT Practice** Suppose every letter in the alphabet has a number value that is equal to its place in the alphabet: the letter A has a value of 1, B a value of 2, and so on. The number value of a word is obtained by adding the values of the letters in the word and then multiplying the sum by the number of letters of the word. Find the number value of the "word" *DFGH*.

A 22 **B** 44 **C** 66 **D** 100 **E** 108

Extra Practice See p. A37.

VOCABULARY

amplitude (p. 368)
angular displacement (p. 352)
angular velocity (p. 352)
central angle (p. 345)
circular arc (p. 345)
compound function (p. 382)
dimensional analysis (p. 353)
frequency (p. 372)
linear velocity (p. 353)
midline (p. 380)
period (p. 359)
periodic (p. 359)
phase shift (p. 378)
principal values (p. 406)
radian (p. 343)
sector (p. 346)
sinusoidal function (p. 388)

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term to best complete each sentence.

1. The (degree, radian) measure of an angle is defined as the length of the corresponding arc on the unit circle.
2. The ratio of the change in the central angle to the time required for the change is known as (angular, linear) velocity.
3. If the values of a function are (different, the same) for each given interval of the domain, the function is said to be periodic.
4. The (amplitude, period) of a function is one-half the difference of the maximum and minimum function values.
5. A central (angle, arc) has a vertex that lies at the center of a circle.
6. A horizontal translation of a trigonometric function is called a (phase, period) shift.
7. The length of a circular arc equals the measure of the radius of the circle times the (degree, radian) measure of the central angle.
8. The period and the (amplitude, frequency) are reciprocals of each other.
9. A function of the form $y = A \sin (k\theta + c) + h$ is a (sinusoidal, compound) function.
10. The values in the (domain, range) of Sine are called principal values.

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 6-1 Change from radian measure to degree measure, and vice versa.

- Change $-\frac{5\pi}{3}$ radians to degree measure.

$$-\frac{5\pi}{3} = \frac{5\pi}{3} \times \frac{180°}{\pi}$$
$$= -300°$$

REVIEW EXERCISES

Change each degree measure to radian measure in terms of π.

11. $60°$ **12.** $-75°$ **13.** $240°$

Change each radian measure to degree measure. Round to the nearest tenth, if necessary.

14. $\frac{5\pi}{6}$ **15.** $-\frac{7\pi}{4}$ **16.** 2.4

Lesson 6-1 Find the length of an arc given the measure of the central angle.

- Given a central angle of $\frac{2\pi}{3}$, find the length of its intercepted arc in a circle of radius 10 inches. Round to the nearest tenth.

$$s = r\theta$$
$$s = 10\left(\frac{2\pi}{3}\right)$$
$$s \approx 20.94395102$$

The length of the arc is about 20.9 inches.

Given the measurement of a central angle, find the length of its intercepted arc in a circle of radius 15 centimeters. Round to the nearest tenth.

17. $\frac{3\pi}{4}$ **18.** $75°$

19. $150°$ **20.** $\frac{\pi}{5}$

Lesson 6-2 Find linear and angular velocity.

- Determine the angular velocity if 5.2 revolutions are completed in 8 seconds. Round to the nearest tenth.

The angular displacement is $5.2 \times 2\pi$ or 10.4π radians.

$$\omega = \frac{\theta}{t}$$
$$\omega = \frac{10.4\pi}{8}$$
$$\omega \approx 4.08407045$$

The angular velocity is about 4.1 radians per second.

Determine each angular displacement in radians. Round to the nearest tenth.

21. 5 revolutions

22. 3.8 revolutions

23. 50.4 revolutions

24. 350 revolutions

Determine each angular velocity. Round to the nearest tenth.

25. 1.8 revolutions in 5 seconds

26. 3.6 revolutions in 2 minutes

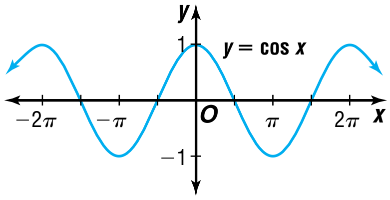
27. 15.4 revolutions in 15 seconds

28. 50 revolutions in 12 minutes

OBJECTIVES AND EXAMPLES

Lesson 6-3 Use the graphs of the sine and cosine functions.

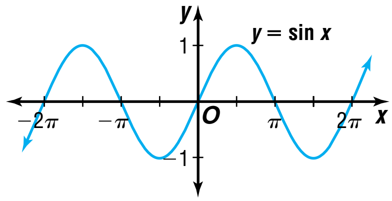
- Find the value of cos $\frac{5\pi}{2}$ by referring to the graph of the cosine function.



$\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$, so $\cos \frac{5\pi}{2} = \cos \frac{\pi}{2}$ or 0.

REVIEW EXERCISES

Find each value by referring to the graph of the cosine function shown at the left or sine function shown below.



29. $\cos 5\pi$

30. $\sin 13\pi$

31. $\sin \frac{9\pi}{2}$

32. $\cos\left(-\frac{7\pi}{2}\right)$

Lesson 6-4 Find the amplitude and period for sine and cosine functions.

- State the amplitude and period for $y = -\frac{3}{4}\cos 2\theta$.

The amplitude of $y = A\cos k\theta$ is $|A|$.
Since $A = -\frac{3}{4}$, the amplitude is $\left|-\frac{3}{4}\right|$ or $\frac{3}{4}$.
Since $k = 2$, the period is $\frac{2\pi}{2}$ or π.

State the amplitude and period for each function. Then graph each function.

33. $y = 4\cos 2\theta$

34. $y = 0.5\sin 4\theta$

35. $y = -\frac{1}{3}\cos\frac{\theta}{2}$

Lesson 6-5 Write equations of sine and cosine functions, given the amplitude, period, phase shift, and vertical translation.

- Write an equation of a cosine function with an amplitude 2, period 2π, phase shift $-\pi$, and vertical shift 2.

***A*:** $|A| = 2$, so $A = 2$ or -2.

***k*:** $\frac{2\pi}{k} = 2\pi$, so $k = 1$.

***c*:** $-\frac{c}{k} = -\pi$, so $-c = -\pi$ or $c = \pi$.

***h*:** $h = 2$

Substituting into $y = A\sin(k\theta + c) + h$, the possible equations are
$y = \pm 2\cos(\theta + \pi) + 2$.

36. Write an equation of a sine function with an amplitude 4, period $\frac{\pi}{2}$, phase shift -2π, and vertical shift -1.

37. Write an equation of a sine function with an amplitude 0.5, period π, phase shift $\frac{\pi}{3}$, and vertical shift 3.

38. Write an equation of a cosine function with an amplitude $\frac{3}{4}$, period $\frac{\pi}{4}$, phase shift 0, and vertical shift 5.

OBJECTIVES AND EXAMPLES

Lesson 6-6 Use sinusoidal functions to solve problems.

- A sinsusoidal function can be any function of the form

  $y = A \sin (k\theta + c) + h$ or

  $y = A \cos (k\theta + c) + h$.

REVIEW EXERCISES

Suppose a person's blood pressure oscillates between the two numbers given. If the heart beats once every second, write a sine function that models this person's blood pressure.

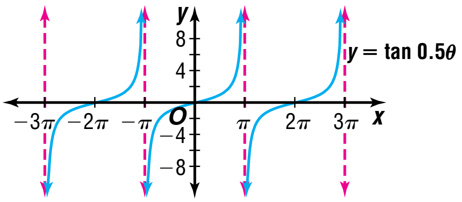
39. 120 and 80

40. 130 and 100

Lesson 6-7 Graph tangent, cotangent, secant, and cosecant functions.

- Graph $y = \tan 0.5\theta$.

  The period of this function is 2π. The phase shift is 0, and the vertical shift is 0.



Graph each function.

41. $y = \frac{1}{3} \csc \theta$

42. $y = 2 \tan \left(3\theta + \frac{\pi}{2}\right)$

43. $y = \sec \theta + 4$

44. $y = \tan \theta - 2$

Lesson 6-8 Find the principal values of inverse trigonometric functions.

- Find $\cos (\text{Tan}^{-1} 1)$.

  Let $\alpha = \text{Tan}^{-1} 1$.

  $\text{Tan}\ \alpha = 1$

  $\alpha = \frac{\pi}{4}$

  $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Find each value.

45. $\text{Arctan} \left(-1\right)$

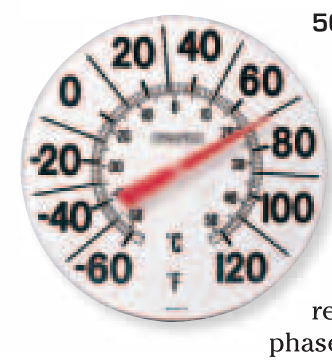
46. $\text{Sin}^{-1} 1$

47. $\text{Cos}^{-1} \left(\tan \frac{\pi}{4}\right)$

48. $\sin \left(\text{Sin}^{-1} \frac{\sqrt{3}}{2}\right)$

49. $\cos \left(\text{Arctan} \sqrt{3} + \text{Arcsin} \frac{1}{2}\right)$

APPLICATIONS AND PROBLEM SOLVING



50. **Meteorology** The mean average temperature in a certain town is 64°F. The temperature fluctuates 11.5° above and below the mean temperature. If $t = 1$ represents January, the phase shift of the sine function is 3. *(Lesson 6-6)*

   a. Write a model for the average monthly temperature in the town.

   b. According to your model, what is the average temperature in April?

   c. According to your model, what is the average temperature in July?

51. **Physics** The strength of a magnetic field is called magnetic induction. An equation for *magnetic induction* is $B = \frac{F}{IL \sin \theta}$, where F is a force on a current I which is moving through a wire of length L at an angle θ to the magnetic field. A wire within a magnetic field is 1 meter long and carries a current of 5.0 amperes. The force on the wire is 0.2 newton, and the magnetic induction is 0.04 newton per ampere-meter. What is the angle of the wire to the magnetic field? *(Lesson 6-8)*

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. The area of a circular sector is about 26.2 square inches. What are possible measures for the radius and the central angle of the sector?

2. a. You are given the graph of a cosine function. Explain how you can tell if the graph has been translated. Sketch two graphs as part of your explanation.

   b. You are given the equation of a cosine function. Explain how you can tell if the graph has been translated. Provide two equations as part of your explanation.

Unit 2 *inter*NET Project

THE CYBERCLASSROOM

What Is Your Sine?

- Search the Internet to find web sites that have applications of the sine or cosine function. Find at least three different sources of information.
- Select one of the applications of the sine or cosine function. Use the Internet to find actual data that can be modeled by a graph that resembles the sine or cosine function.
- Draw a sine or cosine model of the data. Write an equation for a sinusoidal function that fits your data.

PORTFOLIO

Choose a trigonometric function you studied in this chapter. Graph your function. Write three expressions whose values can be found using your graph. Find the values of these expressions.

Additional Assessment See p. A61 for Chapter 6 practice test.

Trigonometry Problems

Each ACT exam contains exactly four trigonometry problems. The SAT has none! You'll need to know the trigonometric functions in a right triangle.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Review the reciprocal functions.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Review the graphs of trigonometric functions.

THE PRINCETON REVIEW

TEST-TAKING TIP

Use the memory aid SOH-CAH-TOA. Pronounce it as *so-ca-to-a.*

SOH represents **S**ine (is) **O**pposite (over) **H**ypotenuse

CAH represents **C**osine (is) **A**djacent (over) **H**ypotenuse

TOA represents **T**angent (is) **O**pposite (over) **A**djacent

ACT EXAMPLE

1. If $\sin\theta = \frac{1}{2}$ and $90° < \theta < 180°$, then $\theta = ?$

A 100°

B 120°

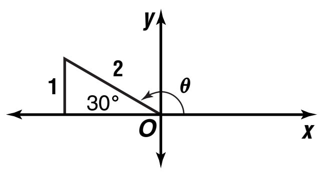
C 130°

D 150°

E 160°

HINT Memorize the sine, cosine, and tangent of special angles 0°, 30°, 45°, 60°, and 90°.

Solution Draw a diagram. Use the quadrant indicated by the size of angle θ.



Recall that the $\sin 30° = \frac{1}{2}$. The angle inside the triangle is 30°. Then $\theta + 30° = 180°$.

If $\theta + 30° = 180°$, then $\theta = 150°$.

The answer is choice **D**.

ACT EXAMPLE

2. What is the least positive value for x where $y = \sin 4x$ reaches its maximum?

A $\frac{\pi}{8}$

B $\frac{\pi}{4}$

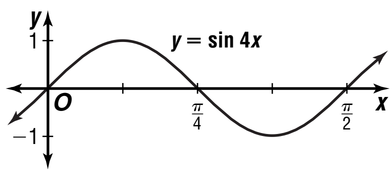
C $\frac{\pi}{2}$

D π

E 2π

HINT Review the graphs of the sine and cosine functions.

Solution The least value for x where $y = \sin x$ reaches its maximum is $\frac{\pi}{2}$. If $4x = \frac{\pi}{2}$, then $x = \frac{\pi}{8}$. The answer is choice **A**.



SAT AND ACT PRACTICE

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. What is $\sin\theta$, if $\tan\theta = \frac{4}{3}$?

A $\frac{3}{4}$ **B** $\frac{4}{5}$

C $\frac{5}{4}$ **D** $\frac{5}{3}$

E $\frac{7}{3}$

2. If the sum of two consecutive odd integers is 56, then the greater integer equals:

A 25 **B** 27 **C** 29

D 31 **E** 33

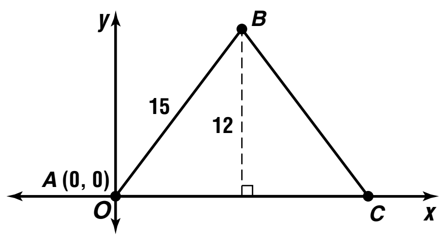
3. For all θ where $\sin\theta - \cos\theta \neq 0$, $\frac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta}$ is equivalent to

A $\sin\theta - \cos\theta$ **B** $\sin\theta + \cos\theta$

C $\tan\theta$ **D** -1

E 1

4. In the figure below, side AB of triangle ABC contains which point?



A (3, 2) **B** (3, 5)

C (4, 6) **D** (4, 10)

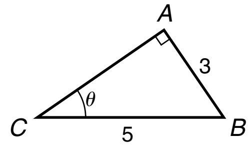
E (6, 8)

5. Which of the following is the sum of both solutions of the equation $x^2 - 2x - 8 = 0$?

A -6 **B** -4 **C** -2

D 2 **E** 6

6. In the figure below, $\angle A$ is a right angle, AB is 3 units long, and BC is 5 units long. If $\angle C = \theta$, what is the value of $\cos\theta$?



A $\frac{3}{5}$ **B** $\frac{3}{4}$ **C** $\frac{4}{5}$ **D** $\frac{5}{4}$ **E** $\frac{5}{3}$

7. The equation $x - 7 = x^2 + y$ represents which conic?

A parabola **B** circle **C** ellipse

D hyperbola **E** line

8. If n is an integer, then which of the following must also be integers?

I. $\frac{16n + 16}{n + 1}$

II. $\frac{16n + 16}{16n}$

III. $\frac{16n^2 + n}{16n}$

A I only **B** II only **C** III only

D I and II **E** II and III

9. For $x > 1$, which expression has a value that is less than 1?

A x^{x-1}

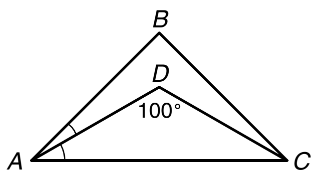
B x^{x+2}

C $(x + 2)^x$

D x^{1-x}

E x^x

10. **Grid-In** In the figure, segment AD bisects $\angle BAC$, and segment DC bisects $\angle BCA$. If the measure of $\angle ADC = 100°$, then what is the measure of $\angle B$?



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