# Chapter 6

# Graphs of Trigonometric Functions

## CHAPTER OBJECTIVES

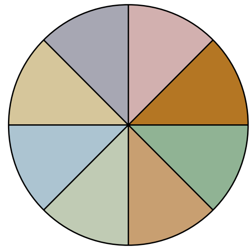
- **Change from radian measure to degree measure, and vice versa.** *(Lesson 6-1)*
- **Find linear and angular velocity.** *(Lesson 6-2)*
- **Use and draw graphs of trigonometric functions and their inverses.** *(Lessons 6-3, 6-4, 6-5, 6-6, 6-7, 6-8)*
- **Find the amplitude, the period, the phase shift, and the vertical shift for trigonometric functions.** *(Lessons 6-4, 6-5, 6-6, 6-7)*
- **Write trigonometric equations to model a given situation.** *(Lessons 6-4, 6-5, 6-6, 6-7)*

# 6-1 Angles and Radian Measure

**OBJECTIVES**
- Change from radian measure to degree measure, and vice versa.
- Find the length of an arc given the measure of the central angle.
- Find the area of a sector.

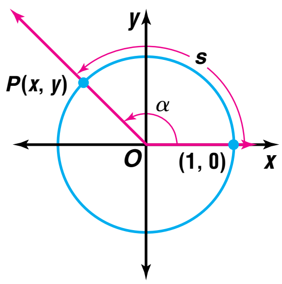
Real World Application

**BUSINESS** Junjira Putiwuthigool owns a business in Changmai, Thailand, that makes ornate umbrellas and fans. Ms. Putiwuthigool has an order for three dozen umbrellas having a diameter of 2 meters. Bamboo slats that support each circular umbrella divide the umbrella into 8 sections or sectors. Each section will be covered with a different color fabric. How much fabric of each color will Ms. Putiwuthigool need to complete the order? *This problem will be solved in Example 6.*



There are many real-world applications, such as the one described above, which can be solved more easily using an angle measure other than the degree. This other unit is called the **radian.**

The definition of radian is based on the concept of the unit circle. Recall that the unit circle is a circle of radius 1 whose center is at the origin of a rectangular coordinate system.




A point $P(x, y)$ is on the unit circle if and only if its distance from the origin is 1. Thus, for each point $P(x, y)$ on the unit circle, the distance from the origin is represented by the following equation.

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 1$$

If each side of this equation is squared, the result is an equation of the unit circle.

$$x^2 + y^2 = 1$$

Consider an angle $\alpha$ in standard position, shown above. Let $P(x, y)$ be the point of intersection of its terminal side with the unit circle. The radian measure of an angle in standard position is defined as the length of the corresponding arc on the unit circle. Thus, the measure of angle $\alpha$ is $s$ radians. Since $C = 2\pi r$, a full revolution correponds to an angle of $2\pi(1)$ or $2\pi$ radians.

There is an important relationship between radian and degree measure. Since an angle of one complete revolution can be represented either by 360° or by $2\pi$ radians, $360° = 2\pi$ radians. Thus, $180° = \pi$ radians, and $90° = \frac{\pi}{2}$ radians.

The following formulas relate degree and radian measures.

**Degree/ Radian Conversion Formulas**

1 radian $= \frac{180}{\pi}$ degrees or about 57.3°

1 degree $= \frac{\pi}{180}$ radians or about 0.017 radian

Angles expressed in radians are often written in terms of $\pi$. The term *radians* is also usually omitted when writing angle measures. However, the degree symbol is always used in this book to express the measure of angles in degrees.

**Example 1** **a. Change 330° to radian measure in terms of $\pi$.**

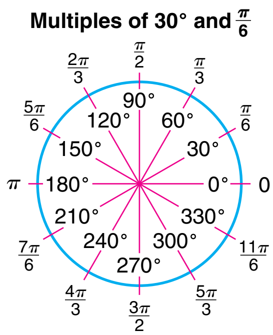
$$330° = 330° \times \frac{\pi}{180°} \quad \textit{1 degree} = \frac{\pi}{180°}$$
$$= \frac{11\pi}{6}$$

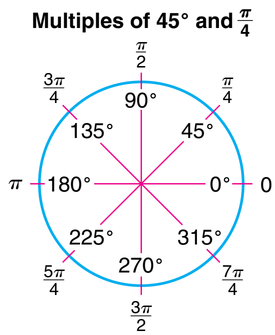
**b. Change $\frac{2\pi}{3}$ radians to degree measure.**

$$\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180°}{\pi} \quad \textit{1 radian} = \frac{180°}{\pi}$$
$$= 120°$$

Angles whose measures are multiples of 30° and 45° are commonly used in trigonometry. These angle measures correspond to radian measures of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, respectively. The diagrams below can help you make these conversions mentally.

*You may want to memorize these radian measures and their degree equivalents to simplify your work in trigonometry.*




These equivalent values are summarized in the chart below.

| **Degrees** | 0 | 30 | 45 | 60 | 90 | 120 | 135 | 150 | 180 | 210 | 225 | 240 | 270 | 300 | 315 | 330 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Radians** | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ |

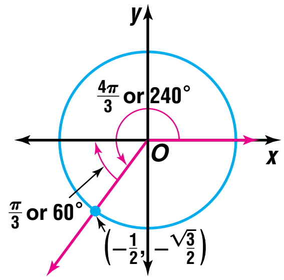
You can use reference angles and the unit circle to determine trigonometric values for angle measures expressed as radians.

## Example 2 Evaluate $\cos \frac{4\pi}{3}$.

> **Look Back**
> You can refer to Lesson 5-3 to review reference angles and unit circles used to determine values of trigonometric functions.

The reference angle for $\frac{4\pi}{3}$ is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.

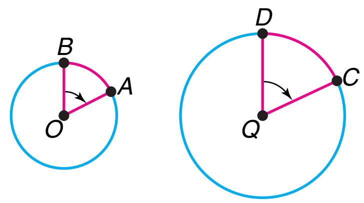
Since $\frac{\pi}{3} = 60°$, the terminal side of the angle intersects the unit circle at a point with coordinates of $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.




Because the terminal side of this angle is in the third quadrant, both coordinates are negative. The point of intersection has coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Therefore, $\cos \frac{4\pi}{3} = -\frac{1}{2}$.

Radian measure can be used to find the length of a **circular arc.** A circular arc is a part of a circle. The arc is often defined by the **central angle** that intercepts it. A central angle of a circle is an angle whose vertex lies at the center of the circle.

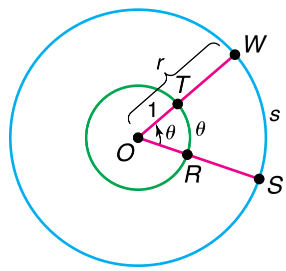



If two central angles in different circles are congruent, the ratio of the lengths of their intercepted arcs is equal to the ratio of the measures of their radii.

For example, given circles $O$ and $Q$, if $\angle O \cong \angle Q$, then $\frac{m\widehat{AB}}{m\widehat{CD}} = \frac{OA}{QC}$.

Let $O$ be the center of two concentric circles, let $r$ be the measure of the radius of the larger circle, and let the smaller circle be a unit circle. A central angle of $\theta$ radians is drawn in the two circles that intercept $\widehat{RT}$ on the unit circle and $\widehat{SW}$ on the other circle. Suppose $\widehat{SW}$ is $s$ units long. $\widehat{RT}$ is $\theta$ units long since it is an arc of a unit circle intercepted by a central angle of $\theta$ radians. Thus, we can write the following proportion.

$$\frac{s}{\theta} = \frac{r}{1} \text{ or } s = r\theta$$




*We say that an arc subtends its central angle.*

**Length of an Arc**

The length of any circular arc $s$ is equal to the product of the measure of the radius of the circle $r$ and the radian measure of the central angle $\theta$ that it subtends.

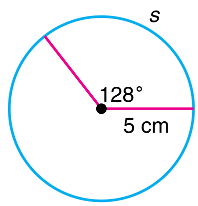
$$s = r\theta$$

**Example 3** **Given a central angle of 128°, find the length of its intercepted arc in a circle of radius 5 centimeters. Round to the nearest tenth.**

First, convert the measure of the central angle from degrees to radians.

$128° = 128° \times \frac{\pi}{180°}$ *1 degree = $\frac{\pi}{180}$*

$= \frac{32}{45}\pi$ or $\frac{32\pi}{45}$




Then, find the length of the arc.

$s = r\theta$

$s = 5\left(\frac{32\pi}{45}\right)$ *$r = 5$, $\theta = \frac{32\pi}{45}$*

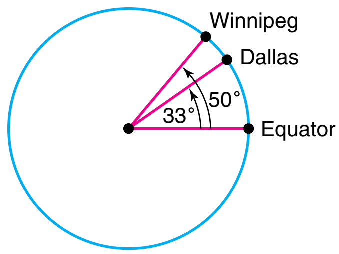
$s \approx 11.17010721$ *Use a calculator.*

The length of the arc is about 11.2 centimeters.

You can use radians to compute distances between two cities that lie on the same longitude line.

**Example 4** Real World Application

**GEOGRAPHY** **Winnipeg, Manitoba, Canada, and Dallas, Texas, lie along the 97° W longitude line. The latitude of Winnipeg is 50° N, and the latitude of Dallas is 33° N. The radius of Earth is about 3960 miles. Find the approximate distance between the two cities.**




The length of the arc between Dallas and Winnipeg is the distance between the two cities. The measure of the central angle subtended by this arc is 50° − 33° or 17°.

$17° = 17° \times \frac{\pi}{180°}$ *1 degree = $\frac{\pi}{180}$*

$= \frac{17\pi}{180}$

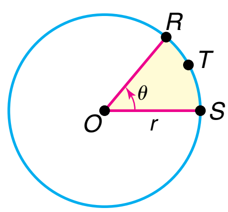
$s = r\theta$

$s = 3960\left(\frac{17\pi}{180}\right)$ *$r = 3960$, $\theta = \frac{17\pi}{180}$*

$s \approx 1174.955652$ *Use a calculator.*

The distance between the two cities is about 1175 miles.

A **sector** of a circle is a region bounded by a central angle and the intercepted arc. For example, the shaded portion in the figure is a sector of circle $O$. The ratio of the area of a sector to the area of a circle is equal to the ratio of its arc length to the circumference.



Let $A$ represent the area of the sector.

$$\frac{A}{\pi r^2} = \frac{\text{length of } \overset{\frown}{RTS}}{2\pi r}$$

$$\frac{A}{\pi r^2} = \frac{r\theta}{2\pi r} \quad \textit{The length of } \overset{\frown}{RTS} \textit{ is } r\theta.$$

$$A = \frac{1}{2}r^2\theta \quad \textit{Solve for A.}$$

**Area of a Circular Sector**

If $\theta$ is the measure of the central angle expressed in radians and $r$ is the measure of the radius of the circle, then the area of the sector, $A$, is as follows.

$$A = \frac{1}{2}r^2\theta$$

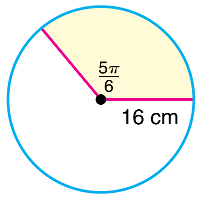
## Examples

**5** **Find the area of a sector if the central angle measures $\frac{5\pi}{6}$ radians and the radius of the circle is 16 centimeters. Round to the nearest tenth.**

$A = \frac{1}{2}r^2\theta$ *Formula for the area of a circular sector*

$A = \frac{1}{2}(16^2)\left(\frac{5\pi}{6}\right)$ $r = 16,\ \theta = \frac{5\pi}{6}$

$A \approx 335.1032164$ *Use a calculator.*




The area of the sector is about 335.1 square centimeters.

**6** **BUSINESS** **Refer to the application at the beginning of the lesson. How much fabric of each color will Ms. Putiwuthigool need to complete the order?**

Real World Application

There are $2\pi$ radians in a complete circle and 8 equal sections or sectors in the umbrella. Therefore, the measure of each central angle is $\frac{2\pi}{8}$ or $\frac{\pi}{4}$ radians. If the diameter of the circle is 2 meters, the radius is 1 meter. Use these values to find the area of each sector.

$A = \frac{1}{2}r^2\theta$

$A = \frac{1}{2}(1^2)\left(\frac{\pi}{4}\right)$ $r = 1,\ \theta = \frac{\pi}{4}$

$A \approx 0.3926990817$ *Use a calculator.*

Since there are 3 dozen or 36 umbrellas, multiply the area of each sector by 36. Ms. Putiwuthigool needs about 14.1 square meters of each color of fabric. *This assumes that the pieces can be cut with no waste and that no extra material is needed for overlapping.*

## CHECK FOR UNDERSTANDING

**Communicating Mathematics**

Read and study the lesson to answer each question.

1. **Draw** a unit circle and a central angle with a measure of $\frac{3\pi}{4}$ radians.
2. **Describe** the angle formed by the hands of a clock at 3:00 in terms of degrees and radians.

3. **Explain** how you could find the radian measure of a central angle subtended by an arc that is 10 inches long in a circle with a radius of 8 inches.

4. **Demonstrate** that if the radius of a circle is doubled and the measure of a central angle remains the same, the length of the arc is doubled and the area of the sector is quadrupled.

**Guided Practice**

**Change each degree measure to radian measure in terms of $\pi$.**

5. $240°$

6. $570°$

**Change each radian measure to degree measure. Round to the nearest tenth, if necessary.**

7. $\frac{3\pi}{2}$

8. $-1.75$

**Evaluate each expression.**

9. $\sin \frac{3\pi}{4}$

10. $\tan \frac{11\pi}{6}$

**Given the measurement of a central angle, find the length of its intercepted arc in a circle of radius 15 inches. Round to the nearest tenth.**

11. $\frac{5\pi}{6}$

12. $77°$

**Find the area of each sector given its central angle $\theta$ and the radius of the circle. Round to the nearest tenth.**

13. $\theta = \frac{2\pi}{3}, r = 1.4$

14. $\theta = 54°, r = 6$

15. **Physics** A pendulum with length of 1.4 meters swings through an angle of 30°. How far does the bob at the end of the pendulum travel as it goes from left to right?

## EXERCISES

**Practice**

**Change each degree measure to radian measure in terms of $\pi$.**

16. $135°$

17. $210°$

18. $300°$

19. $-450°$

20. $-75°$

21. $1250°$

**Change each radian measure to degree measure. Round to the nearest tenth, if necessary.**

22. $\frac{7\pi}{12}$

23. $\frac{11\pi}{3}$

24. $17$

25. $-3.5$

26. $-\frac{\pi}{6.2}$

27. $17.5$

**Evaluate each expression.**

28. $\sin \frac{5\pi}{3}$

29. $\tan \frac{7\pi}{6}$

30. $\cos \frac{5\pi}{4}$

31. $\sin \frac{7\pi}{6}$

32. $\tan \frac{14\pi}{3}$

33. $\cos \left(-\frac{19\pi}{6}\right)$

**Given the measurement of a central angle, find the length of its intercepted arc in a circle of radius 14 centimeters. Round to the nearest tenth.**

**34.** $\frac{2\pi}{3}$ **35.** $\frac{5\pi}{12}$ **36.** $150°$

**37.** $282°$ **38.** $\frac{3\pi}{11}$ **39.** $320°$

**40.** The diameter of a circle is 22 inches. If a central angle measures 78°, find the length of the intercepted arc.

**41.** An arc is 70.7 meters long and is intercepted by a central angle of $\frac{5\pi}{4}$ radians. Find the diameter of the circle.

**42.** An arc is 14.2 centimeters long and is intercepted by a central angle of 60°. What is the radius of the circle?

**Find the area of each sector given its central angle $\theta$ and the radius of the circle. Round to the nearest tenth.**

**43.** $\theta = \frac{5\pi}{12}, r = 10$ **44.** $\theta = 90°, r = 22$ **45.** $\theta = \frac{\pi}{8}, r = 7$

**46.** $\theta = \frac{4\pi}{7}, r = 12.5$ **47.** $\theta = 225°, r = 6$ **48.** $\theta = 82°, r = 7.3$

**49.** A sector has arc length of 6 feet and central angle of 1.2 radians.
  **a.** Find the radius of the circle.
  **b.** Find the area of the sector.

**50.** A sector has a central angle of 135° and arc length of 114 millimeters.
  **a.** Find the radius of the circle.
  **b.** Find the area of the sector.

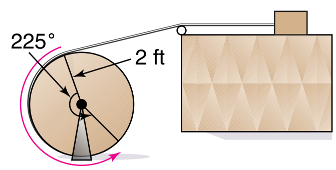
**51.** A sector has area of 15 square inches and central angle of 0.2 radians.
  **a.** Find the radius of the circle.
  **b.** Find the arc length of the sector.

**52.** A sector has area of 15.3 square meters. The radius of the circle is 3 meters.
  **a.** Find the radian measure of the central angle.
  **b.** Find the degree measure of the central angle.
  **c.** Find the arc length of the sector.

## Applications and Problem Solving

Real World Application

**53. Mechanics** A wheel has a radius of 2 feet. As it turns, a cable connected to a box winds onto the wheel.



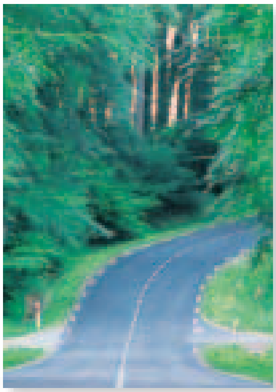

  **a.** How far does the box move if the wheel turns 225° in a counterclockwise direction?
  **b.** Find the number of degrees the wheel must be rotated to move the box 5 feet.

**54. Critical Thinking** Two gears are interconnected. The smaller gear has a radius of 2 inches, and the larger gear has a radius of 8 inches. The smaller gear rotates 330°. Through how many radians does the larger gear rotate?

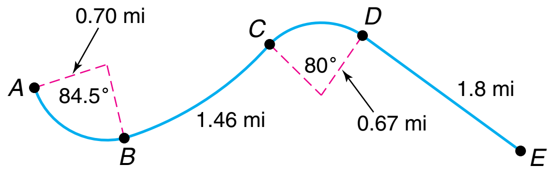
**55. Physics** A pendulum is 22.9 centimeters long, and the bob at the end of the pendulum travels 10.5 centimeters. Find the degree measure of the angle through which the pendulum swings.

**56. Geography** Minneapolis, Minnesota; Arkadelphia, Arkansas; and Alexandria, Louisiana lie on the same longitude line. The latitude of Minneapolis is 45° N, the latitude of Arkadelphia is 34° N, and the latitude of Alexandria is 31° N. The radius of Earth is about 3960 miles.

**a.** Find the approximate distance between Minneapolis and Arkadelphia.
**b.** What is the approximate distance between Minneapolis and Alexandria?
**c.** Find the approximate distance between Arkadelphia and Alexandria.



**57. Civil Engineering** The figure below shows a stretch of roadway where the curves are arcs of circles.



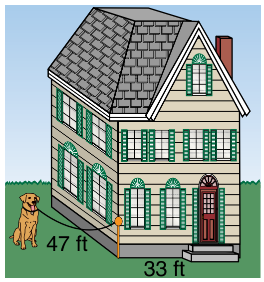

Find the length of the road from point $A$ to point $E$.

**58. Mechanics** A single pulley is being used to pull up a weight. Suppose the diameter of the pulley is $2\frac{1}{2}$ feet.

**a.** How far will the weight rise if the pulley turns 1.5 rotations?
**b.** Find the number of degrees the pulley must be rotated to raise the weight $4\frac{1}{2}$ feet.

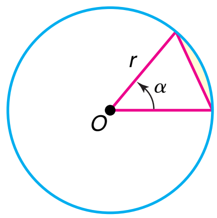
**59. Pet Care** A rectangular house is 33 feet by 47 feet. A dog is placed on a leash that is connected to a pole at the corner of the house.

**a.** If the leash is 15 feet long, find the area the dog has to play.
**b.** If the owner wants the dog to have 750 square feet to play, how long should the owner make the leash?




**60. Biking** Rafael rides his bike 3.5 kilometers. If the radius of the tire on his bike is 32 centimeters, determine the number of radians that a spot on the tire will travel during the trip.

**61. Critical Thinking** A *segment* of a circle is the region bounded by an arc and its chord. Consider any minor arc. If $\alpha$ is the radian measure of the central angle and $r$ is the radius of the circle, write a formula for the area of the segment.

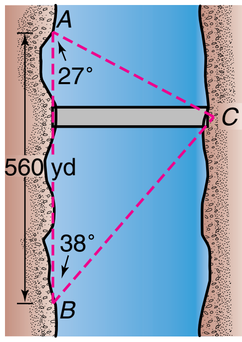



**Mixed Review**

**62.** The lengths of the sides of a triangle are 6 inches, 8 inches, and 12 inches. Find the area of the triangle. *(Lesson 5-8)*

**63.** Determine the number of possible solutions of $\triangle ABC$ if $A = 152°$, $b = 12$, and $a = 10.2$. If solutions exist, solve the triangle. *(Lesson 5-7)*

**64. Surveying** Two surveyors are determining measurements to be used to build a bridge across a canyon. The two surveyors stand 560 yards apart on one side of the canyon and sight a marker $C$ on the other side of the canyon at angles of 27° and 38°. Find the length of the bridge if it is built through point $C$ as shown. *(Lesson 5-6)*




**65.** Suppose $\theta$ is an angle in standard position and $\tan \theta > 0$. State the quadrants in which the terminal side of $\theta$ can lie. *(Lesson 5-3)*

**66. Population** The population for Forsythe County, Georgia, has experienced significant growth in recent years. *(Lesson 4-8)*

| **Year** | 1970 | 1980 | 1990 | 1998 |
|---|---|---|---|---|
| **Population** | 17,000 | 28,000 | 44,000 | 86,000 |

Source: U.S. Census Bureau

a. Write a model that relates the population of Forsythe County as a function of the number of years since 1970.

b. Use the model to predict the population in the year 2020.

**67.** Use the Upper Bound Theorem to find an integral upper bound and the Lower Bound Theorem to find a lower bound of the zeros of $f(x) = x^4 - 3x^3 - 2x^2 + 6x + 10$. *(Lesson 4-5)*

**68.** Use synthetic division to determine if $x + 2$ is a factor of $x^3 + 6x^2 + 12x + 12$. Explain. *(Lesson 4-3)*

**69.** Determine whether the graph of $x^2 + y^2 = 16$ is symmetric with respect to the $x$-axis, the $y$-axis, the line $y = x$, or the line $y = -x$. *(Lesson 3-1)*

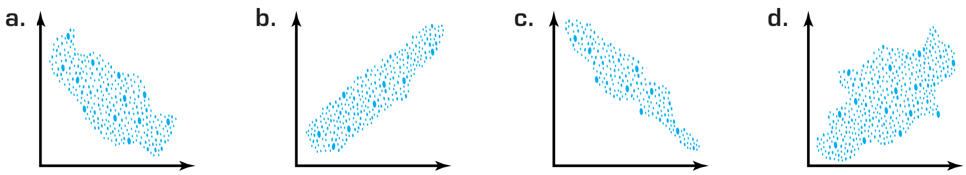
**70.** Solve the system of equations algebraically. *(Lesson 2-2)*

$4x - 2y + 3z = -6$

$3x + 3y - 2z = 2$

$5x - 4y - 3z = -75$

**71.** Which scatter plot shows data that has a strongly positive correlation? *(Lesson 1-6)*




**72. SAT Practice** If $p > 0$ and $q < 0$, which quantity must be positive?

A $p + q$

B $p - q$

C $q - p$

D $p \times q$

E $p \div q$

**Extra Practice** See p. A36.

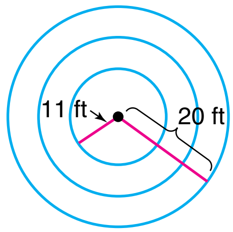
6-2

# Linear and Angular Velocity

**OBJECTIVE**
- Find linear and angular velocity.

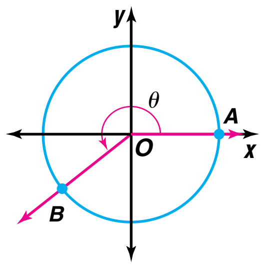
Real World Application

**ENTERTAINMENT** The Children's Museum in Indianapolis, Indiana, houses an antique carousel. The carousel contains three concentric circles of animals. The inner circle of animals is approximately 11 feet from the center, and the outer circle of animals is approximately 20 feet from the center. The carousel makes $2\frac{5}{8}$ rotations per minute. Determine the angular and linear velocities of someone riding an animal in the inner circle and of someone riding an animal in the same row in the outer circle. *This problem will be solved in Examples 3 and 5.*




The carousel is a circular object that turns about an axis through its center. Other examples of objects that rotate about a central axis include Ferris wheels, gears, tires, and compact discs. As the carousel or any other circular object rotates counterclockwise about its center, an object at the edge moves through an angle relative to its starting position known as the **angular displacement,** or angle of rotation.

Consider a circle with its center at the origin of a rectangular coordinate system and point $B$ on the circle rotating counterclockwise. Let the positive $x$-axis, or $\overrightarrow{OA}$, be the initial side of the central angle. The terminal side of the central angle is $\overrightarrow{OB}$. The angular displacement is $\theta$. The measure of $\theta$ changes as $B$ moves around the circle. All points on $\overrightarrow{OB}$ move through the same angle per unit of time.




**Example 1** **Determine the angular displacement in radians of 4.5 revolutions. Round to the nearest tenth.**

Each revolution equals $2\pi$ radians. For 4.5 revolutions, the number of radians is $4.5 \times 2\pi$ or $9\pi$. $9\pi$ radians equals about 28.3 radians.

The ratio of the change in the central angle to the time required for the change is known as **angular velocity.** Angular velocity is usually represented by the lowercase Greek letter $\omega$ (omega).

**Angular Velocity**

If an object moves along a circle during a time of $t$ units, then the angular velocity, $\omega$, is given by

$$\omega = \frac{\theta}{t},$$

where $\theta$ is the angular displacement in radians.

Notice that the angular velocity of a point on a rotating object is not dependent upon the distance from the center of the rotating object.

**Example 2** **Determine the angular velocity if 7.3 revolutions are completed in 5 seconds. Round to the nearest tenth.**

The angular displacement is $7.3 \times 2\pi$ or $14.6\pi$ radians.

$\omega = \frac{\theta}{t}$

$\omega = \frac{14.6\pi}{5}$ *$\theta = 14.6\pi,\ t = 5$*

$\omega \approx 9.173450548$ *Use a calculator.*

The angular velocity is about 9.2 radians per second.

To avoid mistakes when computing with units of measure, you can use a procedure called **dimensional analysis.** In dimensional analyses, unit labels are treated as mathematical factors and can be divided out.

**Example 3** **ENTERTAINMENT** **Refer to the application at the beginning of the lesson. Determine the angular velocity for each rider in radians per second.**

Real World Application

The carousel makes $2\frac{5}{8}$ or 2.625 revolutions per minute. Convert revolutions per minute to radians per second.

$$\frac{2.625 \text{ revolutions}}{1 \text{ minute}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} \times \frac{2\pi \text{ radians}}{1 \text{ revolution}} \approx 0.275 \text{ radian per second}$$

Each rider has an angular velocity of about 0.275 radian per second.

The carousel riders have the same angular velocity. However, the rider in the outer circle must travel a greater distance than the one in the inner circle. The arc length formula can be used to find the relationship between the linear and angular velocities of an object moving in a circular path. If the object moves with constant **linear velocity** ($v$) for a period of time ($t$), the distance ($s$) it travels is given by the formula $s = vt$. Thus, the linear velocity is $v = \frac{s}{t}$.

As the object moves along the circular path, the radius $r$ forms a central angle of measure $\theta$. Since the length of the arc is $s = r\theta$, the following is true.

$s = r\theta$

$\frac{s}{t} = \frac{r\theta}{t}$ *Divide each side by t.*

$v = r\frac{\theta}{t}$ *Replace $\frac{s}{t}$ with v.*

**Linear Velocity**

If an object moves along a circle of radius of $r$ units, then its linear velocity, $v$ is given by

$$v = r\frac{\theta}{t},$$

where $\frac{\theta}{t}$ represents the angular velocity in radians per unit of time.

Since $\omega = \frac{\theta}{t}$, the formula for linear velocity can also be written as $v = r\omega$.

**Examples**

**4** **Determine the linear velocity of a point rotating at an angular velocity of $17\pi$ radians per second at a distance of 5 centimeters from the center of the rotating object. Round to the nearest tenth.**

$v = r\omega$

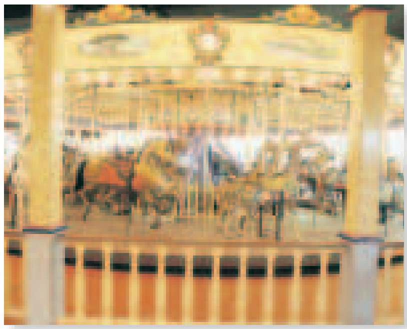
$v = 5(17\pi)$ *$r = 5, \omega = 17\pi$*

$v \approx 267.0353756$ *Use a calculator.*

The linear velocity is about 267.0 centimeters per second.

**5** **ENTERTAINMENT** **Refer to the application at the beginning of the lesson. Determine the linear velocity for each rider.**

From Example 3, you know that the angular velocity is about 0.275 radian per second. Use this number to find the linear velocity for each rider.



**Rider on the Inner Circle**

$v = r\omega$

$v \approx 11(0.275)$ *$r = 11, \omega = 0.275$*

$v \approx 3.025$

**Rider on the Outer Circle**

$v = r\omega$

$v \approx 20(0.275)$ *$r = 20, \omega = 0.275$*

$v \approx 5.5$

The linear velocity of the rider on the inner circle is about 3.025 feet per second, and the linear velocity of the rider on the outer circle is about 5.5 feet per second.

**6** **CAR RACING** **The tires on a race car have a diameter of 30 inches. If the tires are turning at a rate of 2000 revolutions per minute, determine the race car's speed in miles per hour (mph).**

If the diameter is 30 inches, the radius is $\frac{1}{2} \times 30$ or 15 inches. This measure needs to be written in miles. The rate needs to be written in hours.

$$v = \underbrace{15\ \cancel{\text{in.}} \times \frac{1\ \cancel{\text{ft}}}{12\ \cancel{\text{in.}}} \times \frac{1\ \text{mi}}{5280\ \cancel{\text{ft}}}}_{r} \times \underbrace{\frac{2000\ \cancel{\text{rev}}}{1\ \cancel{\text{min}}} \times \frac{2\pi}{1\ \cancel{\text{rev}}} \times \frac{60\ \cancel{\text{min}}}{1\ \text{h}}}_{\omega}$$

$v \approx 178.4995826$ mph *Use a calculator.*

The speed of the race car is about 178.5 miles per hour.

## Check for Understanding

**Communicating Mathematics**

**Read and study the lesson to answer each question.**

1. **Draw** a circle and represent an angular displacement of $3\pi$ radians.
2. **Write** an expression that could be used to change 5 revolutions per minute to radians per second.
3. **Compare and contrast** linear and angular velocity.
4. **Explain** how two people on a rotating carousel can have the same angular velocity but different linear velocity.
5. **Show** that when the radius of a circle is doubled, the angular velocity remains the same and the linear velocity of a point on the circle is doubled.

**Guided Practice**

**Determine each angular displacement in radians. Round to the nearest tenth.**

6. 5.8 revolutions
7. 710 revolutions

**Determine each angular velocity. Round to the nearest tenth.**

8. 3.2 revolutions in 7 seconds
9. 700 revolutions in 15 minutes

**Determine the linear velocity of a point rotating at the given angular velocity at a distance $r$ from the center of the rotating object. Round to the nearest tenth.**

10. $\omega = 36$ radians per second, $r = 12$ inches
11. $\omega = 5\pi$ radians per minute, $r = 7$ meters

12. **Space** A geosynchronous equatorial orbiting (GEO) satellite orbits 22,300 miles above the equator of Earth. It completes one full revolution each 24 hours. Assume Earth's radius is 3960 miles.
    a. How far will the GEO satellite travel in one day?
    b. What is the satellite's linear velocity in miles per hour?

## Exercises

**Practice**

**Determine each angular displacement in radians. Round to the nearest tenth.**

13. 3 revolutions
14. 2.7 revolutions
15. 13.2 revolutions
16. 15.4 revolutions
17. 60.7 revolutions
18. 3900 revolutions

**Determine each angular velocity. Round to the nearest tenth.**

19. 1.8 revolutions in 9 seconds
20. 3.5 revolutions in 3 minutes
21. 17.2 revolutions in 12 seconds
22. 28.4 revolutions in 19 seconds
23. 100 revolutions in 16 minutes
24. 122.6 revolutions in 27 minutes

25. A Ferris wheel rotates one revolution every 50 seconds. What is its angular velocity in radians per second?
26. A clothes dryer is rotating at 500 revolutions per minute. Determine its angular velocity in radians per second.

27. Change 85 radians per second to revolutions per minute (rpm).

**Determine the linear velocity of a point rotating at the given angular velocity at a distance *r* from the center of the rotating object. Round to the nearest tenth.**

28. $\omega = 16.6$ radians per second, $r = 8$ centimeters

29. $\omega = 27.4$ radians per second, $r = 4$ feet

30. $\omega = 6.1\pi$ radians per minute, $r = 1.8$ meters

31. $\omega = 75.3\pi$ radians per second, $r = 17$ inches

32. $\omega = 805.6$ radians per minute, $r = 39$ inches

33. $\omega = 64.5\pi$ radians per minute, $r = 88.9$ millimeters

34. A pulley is turned 120° per second.
   a. Find the number of revolutions per minute (rpm).
   b. If the radius of the pulley is 5 inches, find the linear velocity in inches per second.

35. Consider the tip of each hand of a clock. Find the linear velocity in millimeters per second for each hand.
   a. second hand which is 30 millimeters
   b. minute hand which is 27 millimeters long
   c. hour hand which is 18 millimeters long

## Applications and Problem Solving

Real World Application

36. **Entertainment** The diameter of a Ferris wheel is 80 feet.
   a. If the Ferris wheel makes one revolution every 45 seconds, find the linear velocity of a person riding in the Ferris wheel.
   b. Suppose the linear velocity of a person riding in the Ferris wheel is 8 feet per second. What is the time for one revolution of the Ferris wheel?

37. **Entertainment** The Kit Carson County Carousel makes 3 revolutions per minute.
   a. Find the linear velocity in feet per second of someone riding a horse that is $22\frac{1}{2}$ feet from the center.
   b. The linear velocity of the person on the inside of the carousel is 3.1 feet per second. How far is the person from the center of the carousel?
   c. How much faster is the rider on the outside going than the rider on the inside?

38. **Critical Thinking** Two children are playing on the seesaw. The lighter child is 9 feet from the fulcrum, and the heavier child is 6 feet from the fulcrum. As the lighter child goes from the ground to the highest point, she travels through an angle of 35° in $\frac{1}{2}$ second.
   a. Find the angular velocity of each child.
   b. What is the linear velocity of each child?

39. **Bicycling** A bicycle wheel is 30 inches in diameter.
   a. To the nearest revolution, how many times will the wheel turn if the bicycle is ridden for 3 miles?
   b. Suppose the wheel turns at a constant rate of 2.75 revolutions per second. What is the linear speed in miles per hour of a point on the tire?

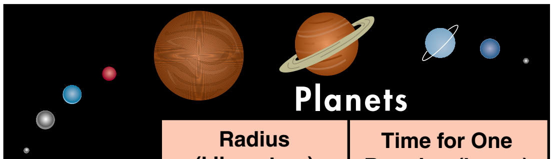
**interNET CONNECTION**

**Research**
For information about the other planets, visit **www.amc.glencoe.com**

**40. Space** The radii and times needed to complete one rotation for the four planets closest to the sun are given at the right.

**a.** Find the linear velocity of a point on each planet's equator.

**b.** Compare the linear velocity of a point on the equator of Mars with a point on the equator of Earth.

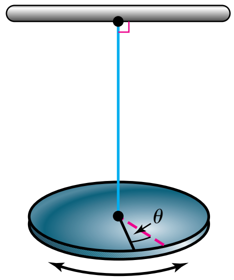


**Planets**

| | Radius (kilometers) | Time for One Rotation (hours) |
|---|---|---|
| Mercury | 2440 | 1407.6 |
| Venus | 6052 | 5832.5 |
| Earth | 6356 | 23.935 |
| Mars | 3375 | 24.623 |

Source: NASA

**41. Physics** A torsion pendulum is an object suspended by a wire or rod so that its plane of rotation is horizontal and it rotates back and forth around the wire without losing energy. Suppose that the pendulum is rotated $\theta_m$ radians and released. Then the angular displacement $\theta$ at time $t$ is $\theta = \theta_m \cos \omega t$, where $\omega$ is the angular frequency in radians per second. Suppose the angular frequency of a certain torsion pendulum is $\pi$ radians per second and its initial rotation is $\frac{\pi}{4}$ radians.




**a.** Write the equation for the angular displacement of the pendulum.

**b.** What are the first two values of $t$ for which the angular displacement of the pendulum is 0?

**42. Space** Low Earth orbiting (LEO) satellites orbit between 200 and 500 miles above Earth. In order to keep the satellites at a constant distance from Earth, they must maintain a speed of 17,000 miles per hour. Assume Earth's radius is 3960 miles.

**a.** Find the angular velocity needed to maintain a LEO satellite at 200 miles above Earth.

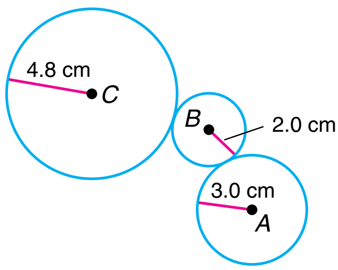
**b.** How far above Earth is a LEO with an angular velocity of 4 radians per hour?

**c.** Describe the angular velocity of any LEO satellite.

**43. Critical Thinking** The figure at the right is a side view of three rollers that are tangent to one another.

**a.** If roller $A$ turns counterclockwise, in which directions do rollers $B$ and $C$ turn?

**b.** If roller $A$ turns at 120 revolutions per minute, how many revolutions per minute do rollers $B$ and $C$ turn?




**Mixed Review**

**44.** Find the area of a sector if the central angle measures 105° and the radius of the circle is 7.2 centimeters. *(Lesson 6-1)*

**45. Geometry** Find the area of a regular pentagon that is inscribed in a circle with a diameter of 7.3 centimeters. *(Lesson 5-4)*

**Extra Practice** See p. A36.



CONTENTS

46. Write 35° 20′ 55″ as a decimal to the nearest thousandth. *(Lesson 5-1)*

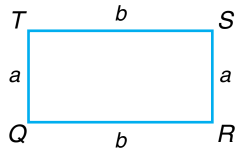
47. Solve $10 + \sqrt{k - 5} = 8$. *(Lesson 4-7)*

48. Write a polynomial equation of least degree with roots $-4$, $3\boldsymbol{i}$, and $-3\boldsymbol{i}$. *(Lesson 4-1)*

49. Graph $y > x^3 + 1$. *(Lesson 3-3)*

50. Write the slope-intercept form of the equation of the line through points at (8, 5) and (−6, 0). *(Lesson 1-4)*

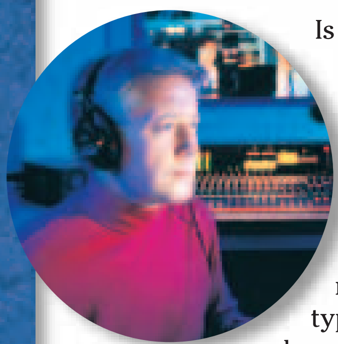
51. **SAT/ACT Practice** The perimeter of rectangle $QRST$ is $p$, and $a = \frac{3}{4}b$. Find the value of $b$ in terms of $p$.




**A** $\frac{p}{7}$ **B** $\frac{4p}{7}$ **C** $\frac{7}{p}$ **D** $\frac{2p}{7}$ **E** $\frac{7p}{4}$

# CAREER CHOICES

## Audio Recording Engineer



Is music your forte? Do you enjoy being creative and solving problems? If you answered yes to these questions, you may want to consider a career as an audio recording engineer. This type of engineer is in charge of all the technical aspects of recording music, speech, sound effects, and dialogue.

Some aspects of the career include controlling the recording equipment, tackling technical problems that arise during recording, and communicating with musicians and music producers. You would need to keep up-to-date on the latest recording equipment and technology. The music producer may direct the sounds you produce through use of the equipment, or you may have the opportunity to design and perfect your own sounds for use in production.

### CAREER OVERVIEW

**Degree Preferred:**
two- or four-year degree in audio engineering

**Related Courses:**
mathematics, music, computer science, electronics

**Outlook:**
number of jobs expected to increase at a slower pace than the average through the year 2006

| Sound | Decibels |
|---|---|
| Threshold of Hearing | 0 |
| Average Whisper (4 feet) | 20 |
| Broadcast Studio (no program in progress) | 30 |
| Soft Recorded Music | 36 |
| Normal Conversation (4 feet) | 60 |
| Moderate Discotheque | 90 |
| Personal Stereo | up to 120 |
| Percussion Instruments at a Symphony Concert | up to 130 |
| Rock Concert | up to 140 |

interNET CONNECTION For more information about audio recording engineering visit: **www.amc.glencoe.com**

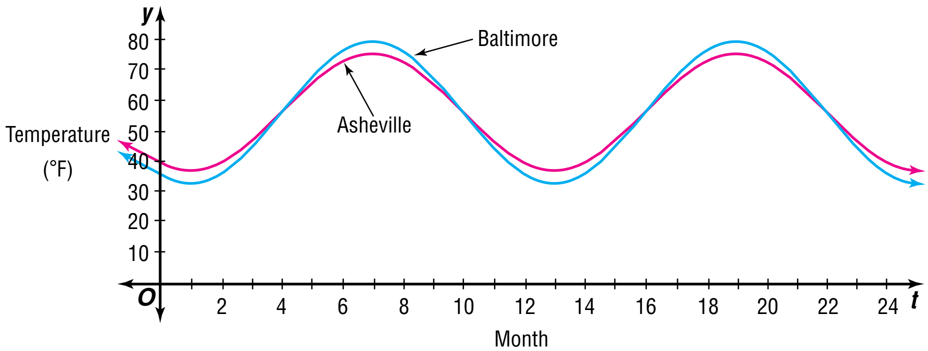
# 6-3 Graphing Sine and Cosine Functions

**OBJECTIVE**

- Use the graphs of the sine and cosine functions.

Real World Application

**METEOROLOGY** The average monthly temperatures for a city demonstrate a repetitious behavior. For cities in the Northern Hemisphere, the average monthly temperatures are usually lowest in January and highest in July. The graph below shows the average monthly temperatures (°F) for Baltimore, Maryland, and Asheville, North Carolina, with January represented by 1.




Model for Baltimore's temperature: $y = 54.4 + 22.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$

Model for Asheville's temperature: $y = 54.5 + 18.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$

In these equations, $t$ denotes the month with January represented by $t = 1$. What is the average temperature for each city for month 13? Which city has the greater fluctuation in temperature?

*These problems will be solved in Example 5.*

Each year, the graph for Baltimore will be about the same. This is also true for Asheville. If the values of a function are the same for each given interval of the domain (in this case, 12 months or 1 year), the function is said to be **periodic.** The interval is the **period** of the function.

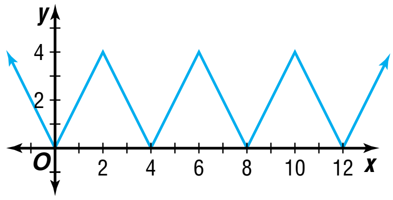
**Periodic Function and Period**

A function is *periodic* if, for some real number $\alpha$, $f(x + \alpha) = f(x)$ for each $x$ in the domain of $f$.

The least positive value of $\alpha$ for which $f(x) = f(x + \alpha)$ is the *period* of the function.

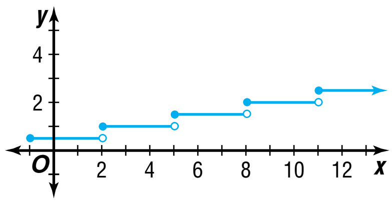
**Example 1** **Determine if each function is periodic. If so, state the period.**

**a.**




The values of the function repeat for each interval of 4 units. The function is periodic, and the period is 4.

**b.**

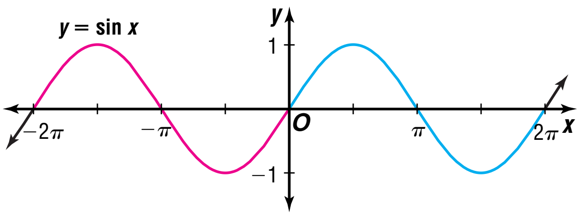



The values of the function do not repeat. The function is not periodic.

Consider the sine function. First evaluate $y = \sin x$ for domain values between $-2\pi$ and $2\pi$ in multiples of $\frac{\pi}{4}$.

| $x$ | $-2\pi$ | $-\frac{7\pi}{4}$ | $-\frac{3\pi}{2}$ | $-\frac{5\pi}{4}$ | $-\pi$ | $-\frac{3\pi}{4}$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sin x$ | 0 | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{2}}{2}$ | 0 | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-\frac{\sqrt{2}}{2}$ | 0 | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{2}}{2}$ | 0 | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-\frac{\sqrt{2}}{2}$ | 0 |

To graph $y = \sin x$, plot the coordinate pairs from the table and connect them to form a smooth curve. Notice that the range values for the domain interval $-2\pi < x < 0$ (shown in red) repeat for the domain interval between $0 < x < 2\pi$ (shown in blue). The sine function is a periodic function.




By studying the graph and its repeating pattern, you can determine the following properties of the graph of the sine function.

**Properties of the Graph of $y = \sin x$**

1. The period is $2\pi$.
2. The domain is the set of real numbers.
3. The range is the set of real numbers between $-1$ and 1, inclusive.
4. The $x$-intercepts are located at $\pi n$, where $n$ is an integer.
5. The $y$-intercept is 0.
6. The maximum values are $y = 1$ and occur when $x = \frac{\pi}{2} + 2\pi n$, where $n$ is an integer.
7. The minimum values are $y = -1$ and occur when $x = \frac{3\pi}{2} + 2\pi n$, where $n$ is an integer.

## Examples

**2** **Find $\sin \frac{9\pi}{2}$ by referring to the graph of the sine function.**

Because the period of the sine function is $2\pi$ and $\frac{9\pi}{2} > 2\pi$, rewrite $\frac{9\pi}{2}$ as a sum involving $2\pi$.

$$\frac{9\pi}{2} = 4\pi + \frac{\pi}{2}$$

$$= 2\pi(2) + \frac{\pi}{2} \quad \textit{This is a form of } \frac{\pi}{2} + 2\pi n.$$

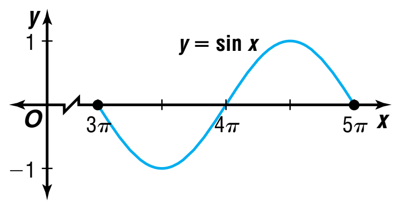
So, $\sin \frac{9\pi}{2} = \sin \frac{\pi}{2}$ or 1.

**3** **Find the values of $\theta$ for which $\sin \theta = 0$ is true.**

Since $\sin \theta = 0$ indicates the $x$-intercepts of the function, $\sin \theta = 0$ if $\theta = n\pi$, where $n$ is any integer.

**4** **Graph $y = \sin x$ for $3\pi \leq x \leq 5\pi$.**

The graph crosses the $x$-axis at $3\pi$, $4\pi$, and $5\pi$. It has its maximum value of 1 at $x = \frac{9\pi}{2}$, and its minimum value of $-1$ at $x = \frac{7\pi}{2}$.
Use this information to sketch the graph.

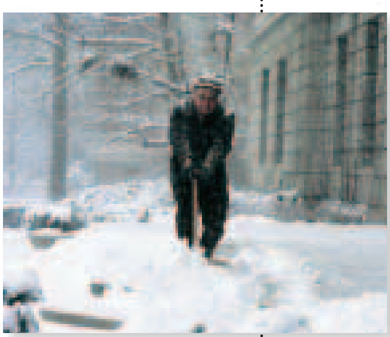



**5** **METEOROLOGY** **Refer to the application at the beginning of the lesson.**

Real World Application

**a. What is the average temperature for each city for month 13?**
Month 13 is January of the second year. To find the average temperature of this month, substitute this value into each equation.

| **Baltimore** | **Asheville** |
|---|---|
| $y = 54.4 + 22.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$ | $y = 54.5 + 18.5 \sin\left[\frac{\pi}{6}(t - 4)\right]$ |
| $y = 54.4 + 22.5 \sin\left[\frac{\pi}{6}(13 - 4)\right]$ | $y = 54.5 + 18.5 \sin\left[\frac{\pi}{6}(13 - 4)\right]$ |
| $y = 54.4 + 22.5 \sin \frac{3\pi}{2}$ | $y = 54.5 + 18.5 \sin \frac{3\pi}{2}$ |
| $y = 54.4 + 22.5(-1)$ | $y = 54.5 + 18.5(-1)$ |
| $y = 31.9$ | $y = 36.0$ |

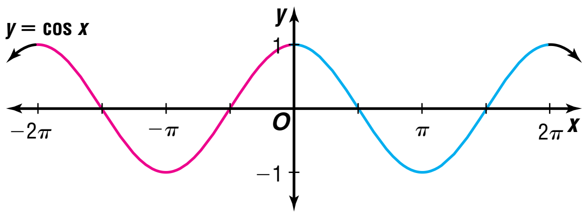


In January, the average temperature for Baltimore is 31.9°, and the average temperature for Asheville is 36.0°.

**b. Which city has the greater fluctuation in temperature? Explain.**
The average temperature for January is lower in Baltimore than in Asheville. The average temperature for July is higher in Baltimore than in Asheville. Therefore, there is a greater fluctuation in temperature in Baltimore than in Asheville.

Now, consider the graph of $y = \cos x$.

| $x$ | $-2\pi$ | $-\frac{7\pi}{4}$ | $-\frac{3\pi}{2}$ | $-\frac{5\pi}{4}$ | $-\pi$ | $-\frac{3\pi}{4}$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\cos x$ | $1$ | $\frac{\sqrt{2}}{2}$ | $0$ | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-\frac{\sqrt{2}}{2}$ | $0$ | $\frac{\sqrt{2}}{2}$ | $1$ | $\frac{\sqrt{2}}{2}$ | $0$ | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-\frac{\sqrt{2}}{2}$ | $0$ | $\frac{\sqrt{2}}{2}$ | $1$ |

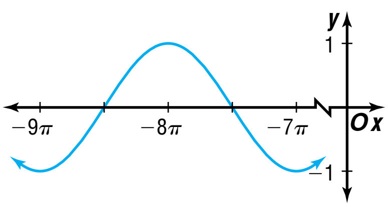



By studying the graph and its repeating pattern, you can determine the following properties of the graph of the cosine function.

**Properties of the Graph of $y = \cos x$**

1. The period is $2\pi$.
2. The domain is the set of real numbers.
3. The range is the set of real numbers between $-1$ and $1$, inclusive.
4. The $x$-intercepts are located at $\frac{\pi}{2} + \pi n$, where $n$ is an integer.
5. The $y$-intercept is 1.
6. The maximum values are $y = 1$ and occur when $x = \pi n$, where $n$ is an even integer.
7. The minimum values are $y = -1$ and occur when $x = \pi n$, where $n$ is an odd integer.

**Example 6** **Determine whether the graph represents $y = \sin x$, $y = \cos x$, or neither.**




The maximum value of 1 occurs when $x = -8\pi$. *maximum of 1 when $x = \pi n \rightarrow \cos x$*

The minimum value of $-1$ occurs at $-9\pi$ and $-7\pi$. *minimum of $-1$ when $x = \pi n \rightarrow \cos x$*

The $x$-intercepts are $-\frac{17\pi}{2}$ and $-\frac{15\pi}{2}$.

These are characteristics of the cosine function. The graph is $y = \cos x$.

## Check for Understanding

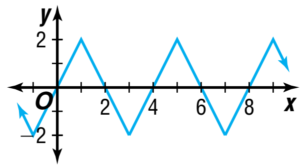
**Communicating Mathematics**

**Read and study the lesson to answer each question.**

1. **Counterexample** Sketch the graph of a periodic function that is neither the sine nor cosine function. State the period of the function.
2. **Name** three values of $x$ that would result in the maximum value for $y = \sin x$.
3. **Explain** why the cosine function is a periodic function.
4. *Math Journal* **Draw** the graphs for the sine function and the cosine function. Compare and contrast the two graphs.

**Guided Practice**

5. **Determine** if the function is periodic. If so, state the period.




**Find each value by referring to the graph of the sine or the cosine function.**

6. $\cos\left(-\frac{\pi}{2}\right)$

7. $\sin \frac{5\pi}{2}$

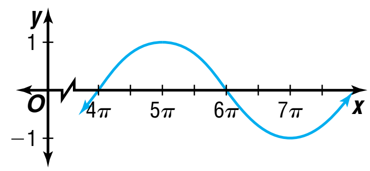
8. Find the values of $\theta$ for which $\sin \theta = -1$ is true.

**Graph each function for the given interval.**

9. $y = \cos x$, $5\pi \leq x \leq 7\pi$

10. $y = \sin x$, $-4\pi \leq x \leq -2\pi$

11. Determine whether the graph represents $y = \sin x$, $y = \cos x$, or neither. Explain.



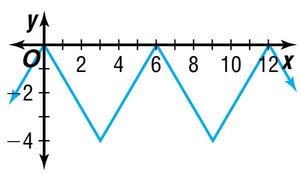

12. **Meteorology** The equation $y = 49 + 28 \sin\left[\frac{\pi}{6}(t - 4)\right]$ models the average monthly temperature for Omaha, Nebraska. In this equation, $t$ denotes the number of months with January represented by 1. Compare the average monthly temperature for April and October.

## Exercises

**Practice**

**Determine if each function is periodic. If so state the period.**

13.

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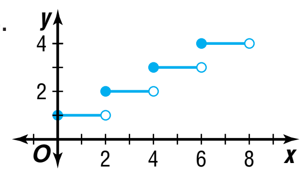


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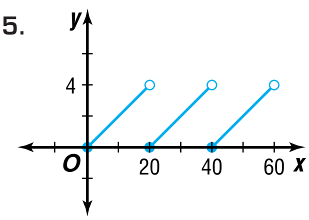



16. $y = |x + 5|$

17. $y = x^2$

18. $y = \frac{1}{x}$

**Find each value by referring to the graph of the sine or the cosine function.**

**19.** $\cos 8\pi$ **20.** $\sin 11\pi$ **21.** $\cos \frac{\pi}{2}$

**22.** $\sin\left(-\frac{3\pi}{2}\right)$ **23.** $\sin \frac{7\pi}{2}$ **24.** $\cos(-3\pi)$

**25.** What is the value of $\sin \pi + \cos \pi$?

**26.** Find the value of $\sin 2\pi - \cos 2\pi$.

**Find the values of $\theta$ for which each equation is true.**

**27.** $\cos \theta = -1$ **28.** $\sin \theta = 1$ **29.** $\cos \theta = 0$

**30.** Under what conditions does $\cos \theta = 1$?

**Graph each function for the given interval.**

**31.** $y = \sin x, -5\pi \leq x \leq -3\pi$

**32.** $y = \cos x, 8\pi \leq x \leq 10\pi$

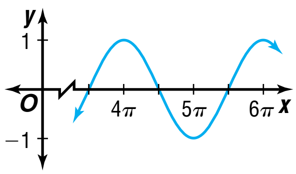
**33.** $y = \cos x, -5\pi \leq x \leq -3\pi$

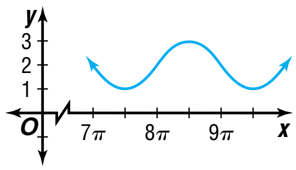
**34.** $y = \sin x, \frac{9\pi}{2} \leq x \leq \frac{13\pi}{2}$

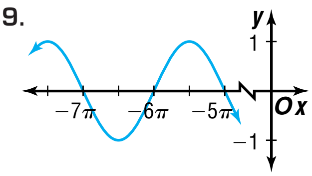
**35.** $y = \cos x, -\frac{7\pi}{2} \leq x \leq -\frac{3\pi}{2}$

**36.** $y = \sin x, \frac{7\pi}{2} \leq x \leq \frac{11\pi}{2}$

**Determine whether each graph is $y = \sin x$, $y = \cos x$, or neither. Explain.**

**37.**



**38.**



**39.**



**40.** Describe a transformation that would change the graph of the sine function to the graph of the cosine function.

**41.** Name any lines of symmetry for the graph of $y = \sin x$.

**42.** Name any lines of symmetry for the graph of $y = \cos x$.

**43.** Use the graph of the sine function to find the values of $\theta$ for which each statement is true.

**a.** $\csc \theta = 1$ **b.** $\csc \theta = -1$ **c.** $\csc \theta$ is undefined.

**44.** Use the graph of the cosine function to find the values of $\theta$ for which each statement is true.

**a.** $\sec \theta = 1$ **b.** $\sec \theta = -1$ **c.** $\sec \theta$ is undefined.

**Graphing Calculator**

**Use a graphing calculator to graph the sine and cosine functions on the same set of axes for $0 \leq x \leq 2\pi$. Use the graphs to find the values of $x$, if any, for which each of the following is true.**

**45.** $\sin x = -\cos x$

**46.** $\sin x \leq \cos x$

**47.** $\sin x \cos x > 1$

**48.** $\sin x \cos x \leq 0$

**49.** $\sin x + \cos x = 1$

**50.** $\sin x - \cos x = 0$

**Applications and Problem Solving**


51. **Meteorology** The equation $y = 43 + 31 \sin\left[\frac{\pi}{6}(t - 4)\right]$ models the average monthly temperatures for Minneapolis, Minnesota. In this equation, $t$ denotes the number of months with January represented by 1.

a. What is the difference between the average monthly temperatures for July and January? What is the relationship between this difference and the coefficient of the sine term?

b. What is the sum of the average monthly temperatures for July and January? What is the relationship between this sum and value of constant term?

52. **Critical Thinking** Consider the graph of $y = 2 \sin x$.

a. What are the $x$-intercepts of the graph?

b. What is the maximum value of $y$?

c. What is the minimum value of $y$?

d. What is the period of the function?

e. Graph the function.

f. How does the 2 in the equation affect the graph?

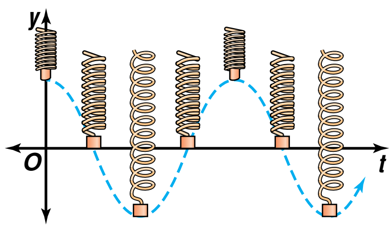
53. **Medicine** The equation $P = 100 + 20 \sin 2\pi t$ models a person's blood pressure $P$ in millimeters of mercury. In this equation, $t$ is time in seconds. The blood pressure oscillates 20 millimeters above and below 100 millimeters, which means that the person's blood pressure is 120 over 80. This function has a period of 1 second, which means that the person's heart beats 60 times a minute.

a. Find the blood pressure at $t = 0$, $t = 0.25$, $t = 0.5$, $t = 0.75$, and $t = 1$.

b. During the first second, when was the blood pressure at a maximum?

c. During the first second, when was the blood pressure at a minimum?

54. **Physics** The motion of a weight on a spring can be described by a modified cosine function. The weight suspended from a spring is at its equilibrium point when it is at rest. When pushed a certain distance above the equilibrium point, the weight oscillates above and below the equilibrium point. The time that it takes for the weight to oscillate from the highest point to the lowest point and back to the highest point is its period. The equation $v = 3.5 \cos\left(t\sqrt{\frac{k}{m}}\right)$ models the vertical displacement $v$ of the weight in relationship to the equilibrium point at any time $t$ if it is initially pushed up 3.5 centimeters. In this equation, $k$ is the elasticity of the spring and $m$ is the mass of the weight.




a. Suppose $k = 19.6$ and $m = 1.99$. Find the vertical displacement after 0.9 second and after 1.7 seconds.

b. When will the weight be at the equilibrium point for the first time?

c. How long will it take the weight to complete one period?

55. **Critical Thinking** Consider the graph of $y = \cos 2x$.

a. What are the $x$-intercepts of the graph?

b. What is the maximum value of $y$?

c. What is the minimum value of $y$?

d. What is the period of the function?

e. Sketch the graph.

56. **Ecology** In predator-prey relationships, the number of animals in each category tends to vary periodically. A certain region has pumas as predators and deer as prey. The equation $P = 500 + 200 \sin [0.4(t - 2)]$ models the number of pumas after $t$ years. The equation $D = 1500 + 400 \sin (0.4t)$ models the number of deer after $t$ years. How many pumas and deer will there be in the region for each value of $t$?

a. $t = 0$ b. $t = 10$ c. $t = 25$

**Mixed Review**

57. **Technology** A computer CD-ROM is rotating at 500 revolutions per minute. Write the angular velocity in radians per second. *(Lesson 6-2)*

58. Change $-1.5$ radians to degree measure. *(Lesson 6-1)*

59. Find the values of $x$ in the interval $0° \leq x \leq 360°$ for which $\sin x = \frac{\sqrt{2}}{2}$. *(Lesson 5-5)*

60. Solve $\frac{2}{x + 2} = \frac{x}{2 - x} + \frac{x^2 + 4}{x^2 - 4}$. *(Lesson 4-6)*

61. Find the number of possible positive real zeros and the number of negative real zeros of $f(x) = 2x^3 + 3x^2 - 11x - 6$. Then determine the rational roots. *(Lesson 4-4)*

62. Use the Remainder Theorem to find the remainder when $x^3 + 2x^2 - 9x + 18$ is divided by $x - 1$. State whether the binomial is a factor of the polynomial. *(Lesson 4-3)*

63. Determine the equations of the vertical and horizontal asymptotes, if any, of $g(x) = \frac{x^2}{x^2 + x}$. *(Lesson 3-7)*

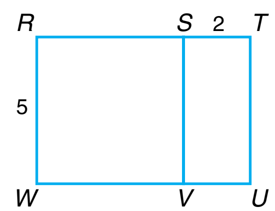
64. Use the graph of the parent function $f(x) = x^3$ to describe the graph of the related function $g(x) = -3x^3$. *(Lesson 3-2)*

65. Find the value of $\begin{vmatrix} -2 & 4 & -1 \\ 1 & -1 & 0 \\ -3 & 4 & 5 \end{vmatrix}$. *(Lesson 2-5)*

66. Use a reflection matrix to find the coordinates of the vertices of $\triangle ABC$ reflected over the $y$-axis for vertices $A$ (3, 2), $B$ (2, −4), and $C$ (1, 6). *(Lesson 2-4)*

67. Graph $x = \frac{3}{2}y$. *(Lesson 1-3)*

68. **SAT/ACT Practice** How much less is the perimeter of square $RSVW$ than the perimeter of rectangle $RTUW$?




A 2 units B 4 units

C 9 units D 12 units

E 20 units


Extra Practice See p. A36.

# HISTORY of MATHEMATICS

## FUNCTIONS

Mathematicians and statisticians use functions to express relationships among sets of numbers. When you use a spreadsheet or a graphing calculator, writing an expression as a function is crucial for calculating values in the spreadsheet or for graphing the function.

**Early Evidence** In about 2000 B.C., the Babylonians used the idea of function in making tables of values for $n$ and $n^3 + n^2$, for $n = 1, 2, \ldots, 30$. Their work indicated that they believed they could show a correspondence between these two sets of values. The following is an example of a Babylonian table.

| $n$ | $n^3 + n^2$ |
|---|---|
| 1 | 2 |
| 2 | 12 |
| ⋮ | ⋮ |
| 30 | ? |

**The Renaissance** In about 1637, **René Descartes** may have been the first person to use the term "function." He defined a function as a power of $x$, such as $x^2$ or $x^3$, where the power was a positive integer. About 55 years later, **Gottfried von Leibniz** defined a function as anything that related to a curve, such as a point on a curve or the slope of a curve. In 1718, **Johann Bernoulli** thought of a function as a relationship between a variable and some constants. Later in that same century, **Leonhard Euler's** notion of a function was an equation or formula with variables and constants. Euler also expanded the notion of function to include not only the written expression, but the graphical representation of the relationship as well. He is credited with the modern standard notation for function, $f(x)$.

Johann Bernoulli

**Modern Era** The 1800s brought **Joseph Lagrange's** idea of function. He limited the meaning of a function to a power series. An example of a power series is $x + x^2 + x^3 + \cdots$, where the three dots indicate that the pattern continues forever. In 1822, **Jean Fourier** determined that any function can be represented by a trigonometric series. **Peter Gustav Dirichlet** used the terminology *y is a function of x* to mean that each first element in the set of ordered pairs is different. Variations of his definition can be found in mathematics textbooks today, including this one.

**Georg Cantor** and others working in the late 1800s and early 1900s are credited with extending the concept of function from ordered pairs of numbers to ordered pairs of elements.

Today engineers like Julia Chang use functions to calculate the efficiency of equipment used in manufacturing. She also uses functions to determine the amount of hazardous chemicals generated during the manufacturing process. She uses spreadsheets to find many values of these functions.

### ACTIVITIES

1. Make a table of values for the Babylonian function, $f(n) = n^3 + n^2$. Use values of $n$ from 1 to 30, inclusive. Then, graph this function using paper and pencil, graphing software, or a graphing calculator. Describe the graph.
2. Research other functions used by notable mathematicians mentioned in this article. You may choose to explore trigonometric series.
3. internet CONNECTION Find out more about personalities referenced in this article and others who contributed to the history of functions. Visit **www.amc.glencoe.com**

# 6-4 Amplitude and Period of Sine and Cosine Functions

**OBJECTIVES**
- Find the amplitude and period for sine and cosine functions.
- Write equations of sine and cosine functions given the amplitude and period.

**BOATING** A signal buoy between the coast of Hilton Head Island, South Carolina, and Savannah, Georgia, bobs up and down in a minor squall. From the highest point to the lowest point, the buoy moves a distance of $3\frac{1}{2}$ feet. It moves from its highest point down to its lowest point and back to its highest point every 14 seconds. Find an equation of the motion for the buoy assuming that it is at its equilibrium point at $t = 0$ and the buoy is on its way down at that time. What is the height of the buoy at 8 seconds and at 17 seconds? *This problem will be solved in Example 5.*

Recall from Chapter 3 that changes to the equation of the parent graph can affect the appearance of the graph by dilating, reflecting, and/or translating the original graph. In this lesson, we will observe the vertical and horizontal expanding and compressing of the parent graphs of the sine and cosine functions.

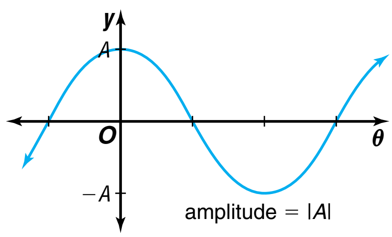
Let's consider an equation of the form $y = A \sin \theta$. We know that the maximum absolute value of $\sin \theta$ is 1. Therefore, for every value of the product of $\sin \theta$ and $A$, the maximum value of $A \sin \theta$ is $|A|$. Similarly, the maximum value of $A \cos \theta$ is $|A|$. The absolute value of $A$ is called the **amplitude** of the functions $y = A \sin \theta$ and $y = A \cos \theta$.

**Amplitude of Sine and Cosine Functions**

The amplitude of the functions $y = A \sin \theta$ and $y = A \cos \theta$ is the absolute value of $A$, or $|A|$.

The amplitude can also be described as the absolute value of one-half the difference of the maximum and minimum function values.

$$|A| = \left|\frac{A - (-A)}{2}\right|$$




**Example 1**

**a. State the amplitude for the function $y = 4 \cos \theta$.**

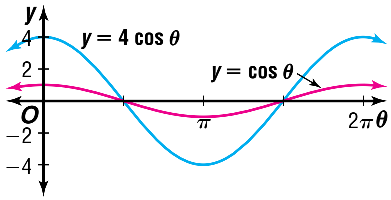
**b. Graph $y = 4 \cos \theta$ and $y = \cos \theta$ on the same set of axes.**

**c. Compare the graphs.**

**a.** According to the definition of amplitude, the amplitude of $y = A \cos \theta$ is $|A|$. So the amplitude of $y = 4 \cos \theta$ is $|4|$ or 4.

**b.** Make a table of values. Then graph the points and draw a smooth curve.

| $\theta$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\cos \theta$ | 1 | $\frac{\sqrt{2}}{2}$ | 0 | $-\frac{\sqrt{2}}{2}$ | $-1$ | $-\frac{\sqrt{2}}{2}$ | 0 | $\frac{\sqrt{2}}{2}$ | 1 |
| $4 \cos \theta$ | 4 | $2\sqrt{2}$ | 0 | $-2\sqrt{2}$ | $-4$ | $-2\sqrt{2}$ | 0 | $2\sqrt{2}$ | 4 |




**c.** The graphs cross the $\theta$-axis at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Also, both functions reach their maximum value at $\theta = 0$ and $\theta = 2\pi$ and their minimum value at $\theta = \pi$. But the maximum and minimum values of the function $y = \cos \theta$ are 1 and $-1$, and the maximum and minimum values of the function $y = 4 \cos \theta$ are 4 and $-4$. The graph of $y = 4 \cos \theta$ is vertically expanded.

## GRAPHING CALCULATOR EXPLORATION

- Select the radian mode.
- Use the domain and range values below to set the viewing window.

$-4.7 \leq x \leq 4.8$, **Xscl: 1** $\quad -3 \leq y \leq 3$, **Yscl: 1**

### TRY THESE

**1.** Graph each function on the same screen.

**a.** $y = \sin x$ **b.** $y = \sin 2x$ **c.** $y = \sin 3x$

### WHAT DO YOU THINK?

**2.** Describe the behavior of the graph of $f(x) = \sin kx$, where $k > 0$, as $k$ increases.

**3.** Make a conjecture about the behavior of the graph of $f(x) = \sin kx$, if $k < 0$. Test your conjecture.

Consider an equation of the form $y = \sin k\theta$, where $k$ is any positive integer. Since the period of the sine function is $2\pi$, the following identity can be developed.

$$y = \sin k\theta$$
$$y = \sin (k\theta + 2\pi) \quad \textit{Definition of periodic function}$$
$$y = \sin k\left(\theta + \frac{2\pi}{k}\right) \quad k\theta + 2\pi = k\left(\theta + \frac{2\pi}{k}\right)$$

Therefore, the period of $y = \sin k\theta$ is $\frac{2\pi}{k}$. Similarly, the period of $y = \cos k\theta$ is $\frac{2\pi}{k}$.

**Period of Sine and Cosine Functions**

The period of the functions $y = \sin k\theta$ and $y = \cos k\theta$ is $\frac{2\pi}{k}$, where $k > 0$.

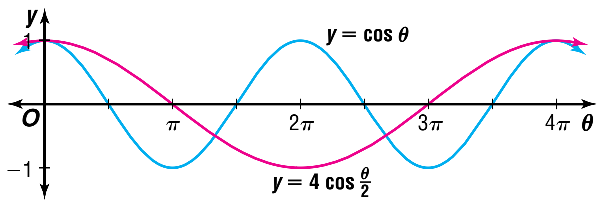
## Example 2

**a. State the period for the function $y = \cos \frac{\theta}{2}$.**

**b. Graph $y = \cos \frac{\theta}{2}$ and $y = \cos \theta$.**

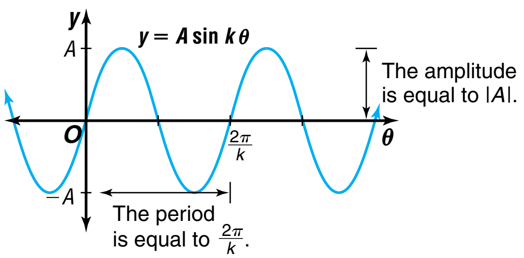
**a.** The definition of the period of $y = \cos k\theta$ is $\frac{2\pi}{k}$. Since $\cos \frac{\theta}{2}$ equals $\cos\left(\frac{1}{2}\theta\right)$, the period is $\frac{2\pi}{\frac{1}{2}}$ or $4\pi$.

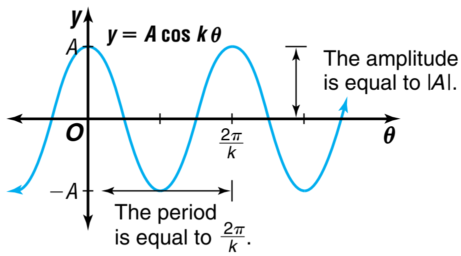
**b.**




*Notice that the graph of $y = \cos \frac{\theta}{2}$ is horizontally expanded.*

The graphs of $y = A \sin k\theta$ and $y = A \cos k\theta$ are shown below.



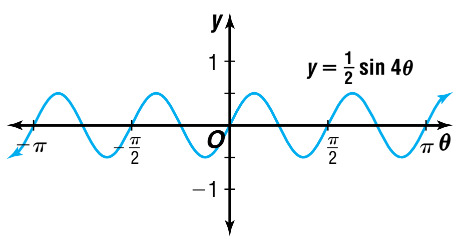

You can use the parent graph of the sine and cosine functions and the amplitude and period to sketch graphs of $y = A \sin k\theta$ and $y = A \cos k\theta$.

## Example 3

**State the amplitude and period for the function $y = \frac{1}{2} \sin 4\theta$. Then graph the function.**

Since $A = \frac{1}{2}$, the amplitude is $\left|\frac{1}{2}\right|$ or $\frac{1}{2}$. Since $k = 4$, the period is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$.

Use the basic shape of the sine function and the amplitude and period to graph the equation.




We can write equations for the sine and cosine functions if we are given the amplitude and period.



CONTENTS

**Example 4** **Write an equation of the cosine function with amplitude 9.8 and period $6\pi$.**

The form of the equation will be $y = A \cos k\theta$. First find the possible values of $A$ for an amplitude of 9.8.

$$|A| = 9.8$$
$$A = 9.8 \text{ or } -9.8$$

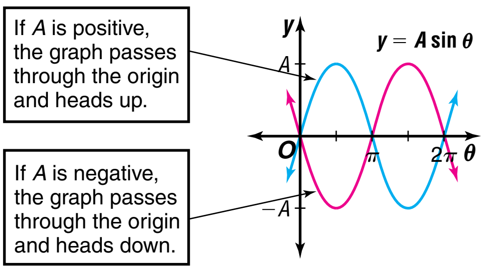
Since there are two values of $A$, two possible equations exist.

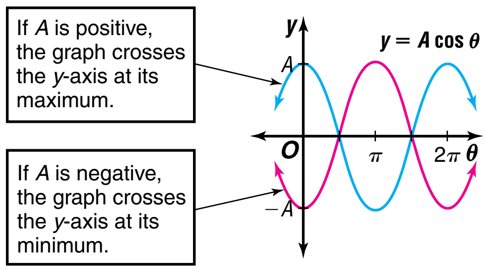
Now find the value of $k$ when the period is $6\pi$.

$$\frac{2\pi}{k} = 6\pi \qquad \textit{The period of a cosine function is } \frac{2\pi}{k}.$$
$$k = \frac{2\pi}{6\pi} \text{ or } \frac{1}{3}$$

The possible equations are $y = 9.8 \cos\left(\frac{1}{3}\theta\right)$ or $y = -9.8 \cos\left(\frac{1}{3}\theta\right)$.

Many real-world situations have periodic characteristics that can be described with the sine and cosine functions. When you are writing an equation to describe a situation, remember the characteristics of the sine and cosine graphs. If you know the function value when $x = 0$ and whether the function is increasing or decreasing, you can choose the appropriate function to write an equation for the situation.




**Example 5** **BOATING** **Refer to the application at the beginning of the lesson.**

Real World Application

**a. Find an equation for the motion of the buoy.**

**b. Determine the height of the buoy at 8 seconds and at 17 seconds.**

**a.** At $t = 0$, the buoy is at equilibrium and is on its way down. This indicates a reflection of the sine function and a negative value of $A$. The general form of the equation will be $y = A \sin kt$, where $A$ is negative and $t$ is the time in seconds.

$$A = -\left(\frac{1}{2} \times 3\frac{1}{2}\right) \qquad \frac{2\pi}{k} = 14$$
$$A = -\frac{7}{4} \text{ or } -1.75 \qquad k = \frac{2\pi}{14} \text{ or } \frac{\pi}{7}$$

An equation for the motion of the buoy is $y = -1.75 \sin \frac{\pi}{7}t$.

**Graphing Calculator Tip**

To find the value of $y$, use a calculator in radian mode.

**b.** Use this equation to find the location of the buoy at the given times.

**At 8 seconds**

$y = -1.75 \sin\left(\frac{\pi}{7} \times 8\right)$

$y \approx 0.7592965435$

At 8 seconds, the buoy is about 0.8 feet above the equilibrium point.

**At 17 seconds**

$y = -1.75 \sin\left(\frac{\pi}{7} \times 17\right)$

$y \approx -1.706123846$

At 17 seconds, the buoy is about 1.7 feet below the equilibrium point.

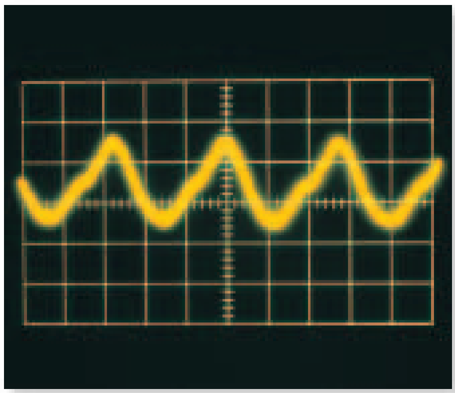
The period represents the amount of time that it takes to complete one cycle. The number of cycles per unit of time is known as the **frequency.** The period (time per cycle) and frequency (cycles per unit of time) are reciprocals of each other.

$$\text{period} = \frac{1}{\text{frequency}} \qquad \text{frequency} = \frac{1}{\text{period}}$$

The *hertz* is a unit of frequency. One hertz equals one cycle per second.

**Example 6** Real World Application

**MUSIC** **Write an equation of the sine function that represents the initial behavior of the vibrations of the note G above middle C having amplitude 0.015 and a frequency of 392 hertz.**



The general form of the equation will be $y = A \sin kt$, where $t$ is the time in seconds. Since the amplitude is 0.015, $A = \pm 0.015$.

The period is the reciprocal of the frequency or $\frac{1}{392}$. Use this value to find $k$.

$\frac{2\pi}{k} = \frac{1}{392}$ *The period* $\frac{2\pi}{k}$ *equals* $\frac{1}{392}$.

$k = 2\pi(392)$ or $784\pi$

One sine function that represents the vibration is $y = 0.015 \sin(784\pi \times t)$.

## CHECK FOR UNDERSTANDING

**Communicating Mathematics**

**Read and study the lesson to answer each question.**

1. **Write** a sine function that has a greater maximum value than the function $y = 4 \sin 2\theta$.
2. **Describe** the relationship between the graphs of $y = 3 \sin \theta$ and $y = -3 \sin \theta$.

3. **Determine** which function has the greatest period.

A. $y = 5 \cos 2\theta$  B. $y = 3 \cos 5\theta$  C. $y = 7 \cos \frac{\theta}{2}$  D. $y = \cos \theta$

4. **Explain** the relationship between period and frequency.

5. *Math Journal* **Draw** the graphs for $y = \cos \theta$, $y = 3 \cos \theta$, and $y = \cos 3\theta$. Compare and contrast the three graphs.

**Guided Practice**

6. State the amplitude for $y = -2.5 \cos \theta$. Then graph the function.

7. State the period for $y = \sin 4\theta$. Then graph the function.

**State the amplitude and period for each function. Then graph each function.**

8. $y = 10 \sin 2\theta$
9. $y = 3 \cos 2\theta$
10. $y = 0.5 \sin \frac{\theta}{6}$
11. $y = -\frac{1}{5} \cos \frac{\theta}{4}$

**Write an equation of the sine function with each amplitude and period.**

12. amplitude = 0.8, period = $\pi$
13. amplitude = 7, period = $\frac{\pi}{3}$

**Write an equation of the cosine function with each amplitude and period.**

14. amplitude = 1.5, period = $5\pi$
15. amplitude = $\frac{3}{4}$, period = 6

16. **Music** Write a sine equation that represents the initial behavior of the vibrations of the note D above middle C having an amplitude of 0.25 and a frequency of 294 hertz.

## EXERCISES

**Practice**

**State the amplitude for each function. Then graph each function.**

17. $y = 2 \sin \theta$
18. $y = -\frac{3}{4} \cos \theta$
19. $y = 1.5 \sin \theta$

**State the period for each function. Then graph each function.**

20. $y = \cos 2\theta$
21. $y = \cos \frac{\theta}{4}$
22. $y = \sin 6\theta$

**State the amplitude and period for each function. Then graph each function.**

23. $y = 5 \cos \theta$
24. $y = -2 \cos 0.5\theta$
25. $y = -\frac{2}{5} \sin 9\theta$
26. $y = 8 \sin 0.5\theta$
27. $y = -3 \sin \frac{\pi}{2}\theta$
28. $y = \frac{2}{3} \cos \frac{3\pi}{7}\theta$
29. $y = 3 \sin 2\theta$
30. $y = 3 \cos 0.5\theta$
31. $y = -\frac{1}{3} \cos 3\theta$
32. $y = \frac{1}{3} \sin \frac{\theta}{3}$
33. $y = -4 \sin \frac{\theta}{2}$
34. $y = -2.5 \cos \frac{\theta}{5}$

35. The equation of the vibrations of the note F above middle C is represented by $y = 0.5 \sin 698\pi t$. Determine the amplitude and period for the function.

**Write an equation of the sine function with each amplitude and period.**

**36.** amplitude $= 0.4$, period $= 10\pi$

**37.** amplitude $= 35.7$, period $= \frac{\pi}{4}$

**38.** amplitude $= \frac{1}{4}$, period $= \frac{\pi}{3}$

**39.** amplitude $= 0.34$, period $= 0.75\pi$

**40.** amplitude $= 4.5$, period $= \frac{5\pi}{4}$

**41.** amplitude $= 16$, period $= 30$

**Write an equation of the cosine function with each amplitude and period.**

**42.** amplitude $= 5$, period $= 2\pi$

**43.** amplitude $= \frac{5}{8}$, period $= \frac{\pi}{7}$

**44.** amplitude $= 7.5$, period $= 6\pi$

**45.** amplitude $= 0.5$, period $= 0.3\pi$

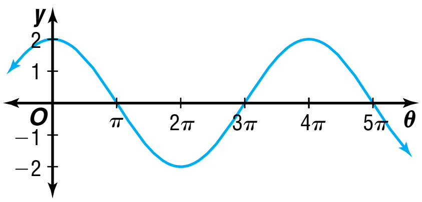
**46.** amplitude $= \frac{2}{5}$, period $= \frac{3}{5}\pi$

**47.** amplitude $= 17.9$, period $= 16$

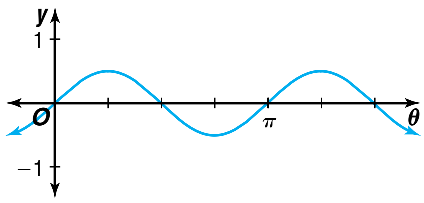
**48.** Write the possible equations of the sine and cosine functions with amplitude 1.5 and period $\frac{\pi}{2}$.

**Write an equation for each graph.**

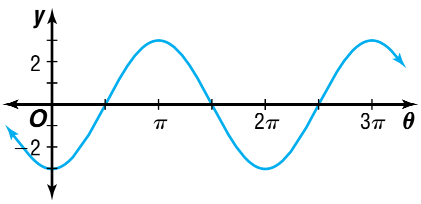
**49.**



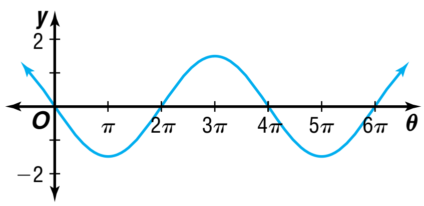

**50.**




**51.**




**52.**




**53.** Write an equation for a sine function with amplitude 3.8 and frequency 120 hertz.

**54.** Write an equation for a cosine function with amplitude 15 and frequency 36 hertz.

**Graphing Calculator**

**55.** Graph these functions on the same screen of a graphing calculator. Compare the graphs.

**a.** $y = \sin x$ **b.** $y = \sin x + 1$ **c.** $y = \sin x + 2$

**Applications and Problem Solving**

56. **Boating** A buoy in the harbor of San Juan, Puerto Rico, bobs up and down. The distance between the highest and lowest point is 3 feet. It moves from its highest point down to its lowest point and back to its highest point every 8 seconds.

a. Find the equation of the motion for the buoy assuming that it is at its equilibrium point at $t = 0$ and the buoy is on its way down at that time.

b. Determine the height of the buoy at 3 seconds.

c. Determine the height of the buoy at 12 seconds.

57. **Critical Thinking** Consider the graph of $y = 2 + \sin \theta$.

a. What is the maximum value of $y$?

b. What is the minimum value of $y$?

c. What is the period of the function?

d. Sketch the graph.

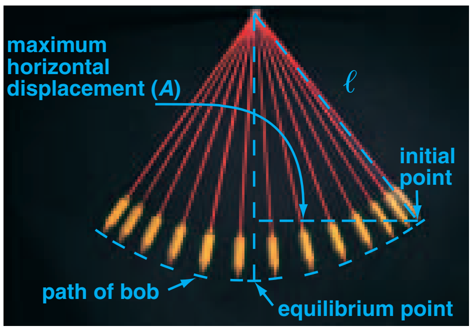
58. **Music** Musical notes are classified by frequency. The note middle C has a frequency of 262 hertz. The note C above middle C has a frequency of 524 hertz. The note C below middle C has a frequency of 131 hertz.

a. Write an equation of the sine function that represents middle C if its amplitude is 0.2.

b. Write an equation of the sine function that represents C above middle C if its amplitude is one half that of middle C.

c. Write an equation of the sine function that represents C below middle C if its amplitude is twice that of middle C.

59. **Physics** For a pendulum, the equation representing the horizontal displacement of the bob is $y = A \cos\left(t\sqrt{\frac{g}{\ell}}\right)$. In this equation, $A$ is the maximum horizontal distance that the bob moves from the equilibrium point, $t$ is the time, $g$ is the acceleration due to gravity, and $\ell$ is the length of the pendulum. The acceleration due to gravity is 9.8 meters per second squared.




a. A pendulum has a length of 6 meters and its bob has a maximum horizontal displacement to the right of 1.5 meters. Write an equation that models the horizontal displacement of the bob if it is at its maximum distance to the right when $t = 0$.

b. Find the location of the bob at 4 seconds.

c. Find the location of the bob at 7.9 seconds.

60. **Critical Thinking** Consider the graph of $y = \cos(\theta + \pi)$.

a. Write an expression for the $x$-intercepts of the graph.

b. What is the $y$-intercept of the graph?

c. What is the period of the function?

d. Sketch the graph.

61. **Physics** Three different weights are suspended from three different springs. Each spring has an elasticity coefficient of 18.5. The equation for the vertical displacement is $y = 1.5 \cos\left(t\sqrt{\frac{k}{m}}\right)$, where $t$ is time, $k$ is the elasticity coefficient, and $m$ is the mass of the weight.

a. The first weight has a mass of 0.4 kilogram. Find the period and frequency of this spring.

b. The second weight has a mass of 0.6 kilogram. Find the period and frequency of this spring.

c. The third weight has a mass of 0.8 kilogram. Find the period and frequency of this spring.

d. As the mass increases, what happens to the period?

e. As the mass increases, what happens to the frequency?

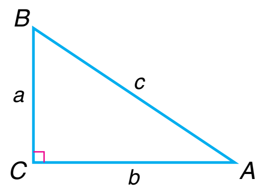
**Mixed Review**

62. Find $\cos\left(-\frac{5\pi}{2}\right)$ by referring to the graph of the cosine function. *(Lesson 6-3)*

63. Determine the angular velocity if 84 revolutions are completed in 6 seconds. *(Lesson 6-2)*

64. Given a central angle of 73°, find the length of its intercepted arc in a circle of radius 9 inches. *(Lesson 6-1)*

65. Solve the triangle if $a = 15.1$ and $b = 19.5$. Round to the nearest tenth. *(Lesson 5-5)*




66. **Physics** The period of a pendulum can be determined by the formula $T = 2\pi\sqrt{\frac{\ell}{g}}$, where $T$ represents the period, $\ell$ represents the length of the pendulum, and $g$ represents the acceleration due to gravity. Determine the length of the pendulum if the pendulum has a period on Earth of 4.1 seconds and the acceleration due to gravity at Earth's surface is 9.8 meters per second squared. *(Lesson 4-7)*

67. Find the discriminant of $3m^2 + 5m + 10 = 0$. Describe the nature of the roots. *(Lesson 4-2)*

68. **Manufacturing** Icon, Inc. manufactures two types of computer graphics cards, Model 28 and Model 74. There are three stations, $A$, $B$, and $C$, on the assembly line. The assembly of a Model 28 graphics card requires 30 minutes at station $A$, 20 minutes at station $B$, and 12 minutes at station $C$. Model 74 requires 15 minutes at station $A$, 30 minutes at station $B$, and 10 minutes at station $C$. Station $A$ can be operated for no more than 4 hours a day, station $B$ can be operated for no more than 6 hours a day, and station $C$ can be operated for no more than 8 hours. *(Lesson 2-7)*

a. If the profit on Model 28 is \$100 and on Model 74 is \$60, how many of each model should be assembled each day to provide maximum profit?

b. What is the maximum daily profit?

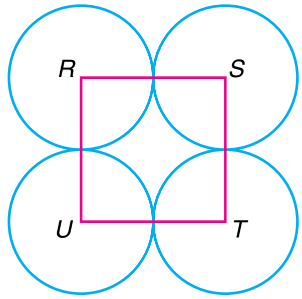
69. Use a reflection matrix to find the coordinates of the vertices of a quadrilateral reflected over the $x$-axis if the coordinates of the vertices of the quadrilateral are located at $(-2, -1)$, $(1, -1)$, $(3, -4)$, and $(-3, -2)$. *(Lesson 2-4)*

70. Graph $g(x) = \begin{cases} -3x \text{ if } x < -2 \\ 2 \text{ if } -2 \leq x < 3. \\ x + 1 \text{ if } x \geq 3 \end{cases}$ *(Lesson 1-7)*

71. **Fund-Raising** The regression equation of a set of data is $y = 14.7x + 140.1$, where $y$ represents the money collected for a fund-raiser and $x$ represents the number of members of the organization. Use the equation to predict the amount of money collected by 20 members. *(Lesson 1-6)*

72. Given that $x$ is an integer, state the relation representing $y = x^2$ and $-4 \leq x \leq -2$ by listing a set of ordered pairs. Then state whether this relation is a function. *(Lesson 1-1)*

73. **SAT/ACT Practice** Points $RSTU$ are the centers of four congruent circles. If the area of square $RSTU$ is 100, what is the sum of the areas of the four circles?




A $25\pi$
B $50\pi$
C $100\pi$
D $200\pi$
E $400\pi$

## MID-CHAPTER QUIZ

1. Change $\frac{5\pi}{6}$ radians to degree measure. (Lesson 6-1)

2. **Mechanics** A pulley with diameter 0.5 meter is being used to lift a box. How far will the box weight rise if the pulley is rotated through an angle of $\frac{5\pi}{3}$ radians? (Lesson 6-1)

3. Find the area of a sector if the central angle measures $\frac{2\pi}{5}$ radians and the radius of the circle is 8 feet. (Lesson 6-1)

4. Determine the angular displacement in radians of 7.8 revolutions. (Lesson 6-2)

5. Determine the angular velocity if 8.6 revolutions are completed in 7 seconds. (Lesson 6-2)

6. Determine the linear velocity of a point rotating at an angular velocity of $8\pi$ radians per second at a distance of 3 meters from the center of the rotating object. (Lesson 6-2)

7. Find $\sin\left(-\frac{7\pi}{2}\right)$ by referring to the graph of the sine function. (Lesson 6-3)

8. Graph $y = \cos x$ for $7\pi \leq x \leq 9\pi$. (Lesson 6-3)

9. State the amplitude and period for the function $y = -7 \cos \frac{\theta}{3}$. Then graph the function. (Lesson 6-4)

10. Find the possible equations of the sine function with amplitude 5 and period $\frac{\pi}{3}$. (Lesson 6-4)

**Extra Practice** See p. A36.

6-5

# Translations of Sine and Cosine Functions

**OBJECTIVES**

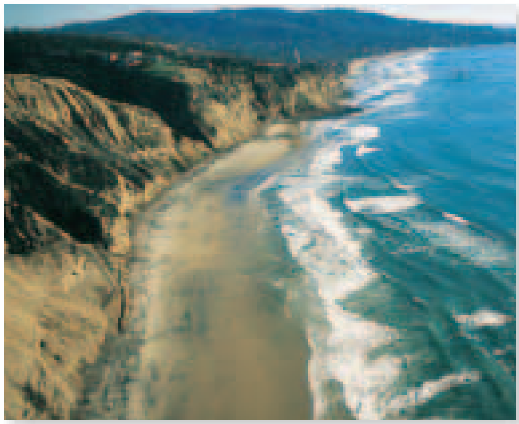
- Find the phase shift and the vertical translation for sine and cosine functions.
- Write the equations of sine and cosine functions given the amplitude, period, phase shift, and vertical translation.
- Graph compound functions.

Real World Application

**TIDES** One day in March in San Diego, California, the first low tide occurred at 1:45 A.M., and the first high tide occurred at 7:44 A.M. Approximately 12 hours and 24 minutes or 12.4 hours after the first low tide occurred, the second low tide occurred. The equation that models these tides is

$h = 2.9 + 2.2 \sin\left(\frac{\pi}{6.2}t - \frac{4.85\pi}{6.2}\right)$,

where $t$ represents the number of hours since midnight and $h$ represents the height of the water. Draw a graph that models the cyclic nature of the tide. *This problem will be solved in Example 4.*



In Chapter 3, you learned that the graph of $y = (x - 2)^2$ is a horizontal translation of the parent graph of $y = x^2$. Similarly, graphs of the sine and cosine functions can be translated horizontally.

## GRAPHING CALCULATOR EXPLORATION

- Select the radian mode.
- Use the domain and range values below to set the viewing window.

$-4.7 \le x \le 4.8$, **Xscl: 1** $\quad -3 \le y \le 3$, **Yscl: 1**

**TRY THESE**

**1.** Graph each function on the same screen.

**a.** $y = \sin x$ **b.** $y = \sin\left(x + \frac{\pi}{4}\right)$

**c.** $y = \sin\left(x + \frac{\pi}{2}\right)$

**WHAT DO YOU THINK?**

**2.** Describe the behavior of the graph of $f(x) = \sin(x + c)$, where $c > 0$, as $c$ increases.

**3.** Make a conjecture about what happens to the graph of $f(x) = \sin(x + c)$ if $c < 0$ and continues to decrease. Test your conjecture.

A horizontal translation or shift of a trigonometric function is called a **phase shift.** Consider the equation of the form $y = A \sin(k\theta + c)$, where $A, k, c \neq 0$. To find a zero of the function, find the value of $\theta$ for which $A \sin(k\theta + c) = 0$. Since $\sin 0 = 0$, solving $k\theta + c = 0$ will yield a zero of the function.

$$k\theta + c = 0$$
$$\theta = -\frac{c}{k} \quad \text{Solve for } \theta.$$

Therefore, $y = 0$ when $\theta = -\frac{c}{k}$. The value of $-\frac{c}{k}$ is the phase shift.

When $c > 0$: The graph of $y = A \sin (k\theta + c)$ is the graph of $y = A \sin k\theta$, shifted $\left|\frac{c}{k}\right|$ to the left.

When $c < 0$: The graph of $y = A \sin (k\theta + c)$ is the graph of $y = A \sin k\theta$, shifted $\left|\frac{c}{k}\right|$ to the right.

**Phase Shift of Sine and Cosine Functions**

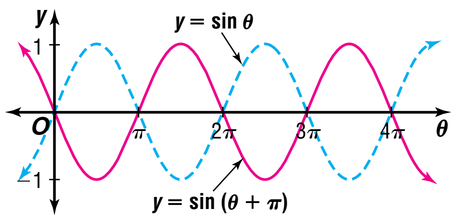
The phase shift of the functions $y = A \sin (k\theta + c)$ and $y = A \cos (k\theta + c)$ is $-\frac{c}{k}$, where $k > 0$.
If $c > 0$, the shift is to the left.
If $c < 0$, the shift is to the right.

**Example 1** **State the phase shift for each function. Then graph the function.**

**a.** $y = \sin (\theta + \pi)$

The phase shift of the function is $-\frac{c}{k}$ or $-\frac{\pi}{1}$, which equals $-\pi$.

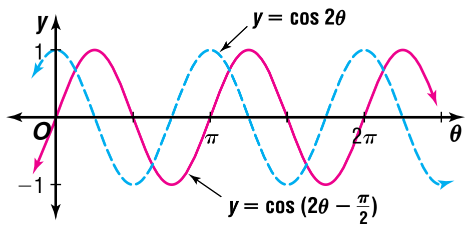
To graph $y = \sin (\theta + \pi)$, consider the graph of $y = \sin \theta$. Graph this function and then shift the graph $-\pi$.




**b.** $y = \cos \left(2\theta - \frac{\pi}{2}\right)$

The phase shift of the function is $-\frac{c}{k}$ or $-\left(\frac{-\frac{\pi}{2}}{2}\right)$, which equals $\frac{\pi}{4}$.

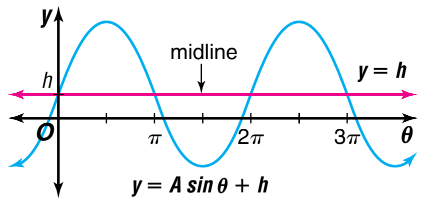
To graph $y = \cos \left(2\theta - \frac{\pi}{2}\right)$, consider the graph of $y = \cos 2\theta$. The graph of $y = \cos 2\theta$ has amplitude of 1 and a period of $\frac{2\pi}{2}$ or $\pi$. Graph this function and then shift the graph $\frac{\pi}{4}$.



In Chapter 3, you also learned that the graph of $y = x^2 - 2$ is a vertical translation of the parent graph of $y = x^2$. Similarly, graphs of the sine and cosine functions can be translated vertically.

When a constant is added to a sine or cosine function, the graph is shifted upward or downward. If $(x, y)$ are the coordinates of $y = \sin x$, then $(x, y + d)$ are the coordinates of $y = \sin x + d$.

A new horizontal axis known as the **midline** becomes the reference line or equilibrium point about which the graph oscillates. For the graph of $y = A \sin \theta + h$, the midline is the graph of $y = h$.

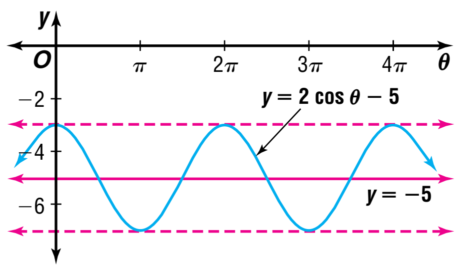



| | |
|---|---|
| **Vertical Shift of Sine and Cosine Functions** | The vertical shift of the functions $y = A \sin (k\theta + c) + h$ and $y = A \cos (k\theta + c) + h$ is $h$. If $h > 0$, the shift is upward. If $h < 0$, the shift is downward. The midline is $y = h$. |

**Example 2** **State the vertical shift and the equation of the midline for the function $y = 2 \cos \theta - 5$. Then graph the function.**

The vertical shift is 5 units downward. The midline is the graph of $y = -5$.

To graph the function, draw the midline, the graph of $y = -5$. Since the amplitude of the function is $|2|$ or 2, draw dashed lines parallel to the midline which are 2 units above and below the midline. That is, $y = -3$ and $y = -7$. Then draw the cosine curve.




In general, use the following steps to graph any sine or cosine function.

| | |
|---|---|
| **Graphing Sine and Cosine Functions** | 1. Determine the vertical shift and graph the midline.<br>2. Determine the amplitude. Use dashed lines to indicate the maximum and minimum values of the function.<br>3. Determine the period of the function and graph the appropriate sine or cosine curve.<br>4. Determine the phase shift and translate the graph accordingly. |

**Example 3** **State the amplitude, period, phase shift, and vertical shift for $y = 4\cos\left(\frac{\theta}{2} + \pi\right) - 6$. Then graph the function.**

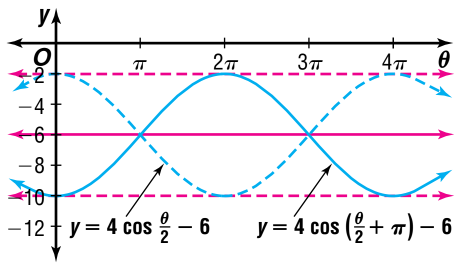
The amplitude is $|4|$ or 4. The period is $\frac{2\pi}{\frac{1}{2}}$ or $4\pi$. The phase shift is $-\frac{\pi}{\frac{1}{2}}$ or $-2\pi$. The vertical shift is $-6$. Using this information, follow the steps for graphing a cosine function.

**Step 1** Draw the midline which is the graph of $y = -6$.

**Step 2** Draw dashed lines parallel to the midline, which are 4 units above and below the midline.

**Step 3** Draw the cosine curve with period of $4\pi$.

**Step 4** Shift the graph $2\pi$ units to the left.




You can use information about amplitude, period, and translations of sine and cosine functions to model real-world applications.

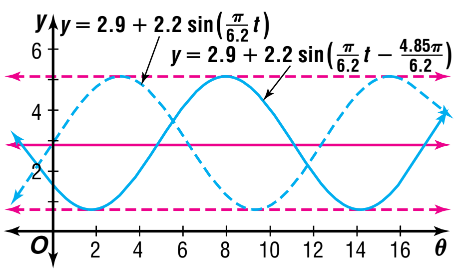
**Example 4** *Real World Application* **TIDES** **Refer to the application at the beginning of the lesson. Draw a graph that models the San Diego tide.**

The vertical shift is 2.9. Draw the midline $y = 2.9$.

The amplitude is $|2.2|$ or 2.2. Draw dashed lines parallel to and 2.2 units above and below the midline.

The period is $\frac{2\pi}{\frac{\pi}{6.2}}$ or 12.4. Draw the sine curve with a period of 12.4.

Shift the graph $-\frac{\frac{-4.85\pi}{6.2}}{\frac{\pi}{6.2}}$ or 4.85 units.




You can write an equation for a trigonometric function if you are given the amplitude, period, phase shift, and vertical shift.

**Example 5** **Write an equation of a sine function with amplitude 4, period $\pi$, phase shift $-\frac{\pi}{8}$, and vertical shift 6.**

The form of the equation will be $y = A \sin (k\theta + c) + h$. Find the values of $A$, $k$, $c$, and $h$.

$A$: $|A| = 4$
$A = 4$ or $-4$

$k$: $\frac{2\pi}{k} = \pi$ *The period is $\pi$.*
$k = 2$

$c$: $-\frac{c}{k} = -\frac{\pi}{8}$ *The phase shift is $-\frac{\pi}{8}$.*
$-\frac{c}{2} = -\frac{\pi}{8}$ *$k = 2$*
$c = \frac{\pi}{4}$

$h$: $h = 6$

Substitute these values into the general equation. The possible equations are $y = 4 \sin \left(2\theta + \frac{\pi}{4}\right) + 6$ and $y = -4 \sin \left(2\theta + \frac{\pi}{4}\right) + 6$.

**Compound functions** may consist of sums or products of trigonometric functions. Compound functions may also include sums and products of trigonometric functions and other functions.

Here are some examples of compound functions.

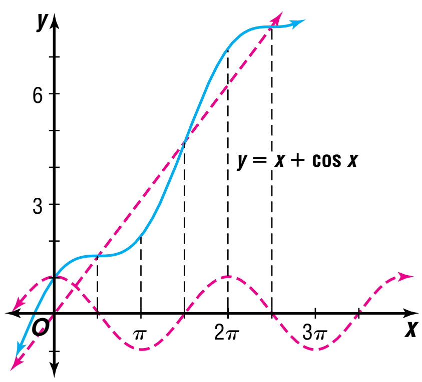
$y = \sin x \cdot \cos x$ *Product of trigonometric functions*
$y = \cos x + x$ *Sum of a trigonometric function and a linear function*

You can graph compound functions involving addition by graphing each function separately on the same coordinate axes and then adding the ordinates. After you find a few of the critical points in this way, you can sketch the rest of the curve of the function of the compound function.

**Example 6** **Graph $y = x + \cos x$.**

First graph $y = \cos x$ and $y = x$ on the same axis. Then add the corresponding ordinates of the function. Finally, sketch the graph.

| $x$ | $\cos x$ | $x + \cos x$ |
|---|---|---|
| 0 | 1 | 1 |
| $\frac{\pi}{2}$ | 0 | $\frac{\pi}{2} + 0 \approx 1.57$ |
| $\pi$ | $-1$ | $\pi - 1 \approx 2.14$ |
| $\frac{3\pi}{2}$ | 0 | $\frac{3\pi}{2} \approx 4.71$ |
| $2\pi$ | 1 | $2\pi + 1 \approx 7.28$ |
| $\frac{5\pi}{2}$ | 0 | $\frac{5\pi}{2} \approx 7.85$ |
| $3\pi$ | $-1$ | $3\pi - 1 \approx 8.42$ |






CONTENTS

## CHECK FOR UNDERSTANDING

**Communicating Mathematics**

**Read and study the lesson to answer each question.**

1. **Compare and contrast** the graphs $y = \sin x + 1$ and $y = \sin(x + 1)$.
2. **Name** the function whose graph is the same as the graph of $y = \cos x$ with a phase shift of $\frac{\pi}{2}$.
3. **Analyze** the function $y = A \sin(k\theta + c) + h$. Which variable could you increase or decrease to have each of the following effects on the graph?
   a. stretch the graph vertically
   b. translate the graph downward vertically
   c. shrink the graph horizontally
   d. translate the graph to the left.
4. **Explain** how to graph $y = \sin x + \cos x$.
5. **You Decide** Marsha and Jamal are graphing $y = \cos\left(\frac{\pi}{6}\theta - \frac{\pi}{2}\right)$. Marsha says that the phase shift of the graph is $\frac{\pi}{2}$. Jamal says that the phase shift is 3. Who is correct? Explain.

**Guided Practice**

6. State the phase shift for $y = 3\cos\left(\theta - \frac{\pi}{2}\right)$. Then graph the function.
7. State the vertical shift and the equation of the midline for $y = \sin 2\theta + 3$. Then graph the function.

**State the amplitude, period, phase shift, and vertical shift for each function. Then graph the function.**

8. $y = 2\sin(2\theta + \pi) - 5$
9. $y = 3 - \frac{1}{2}\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$
10. Write an equation of a sine function with amplitude 20, period 1, phase shift 0, and vertical shift 100.
11. Write an equation of a cosine function with amplitude 0.6, period 12.4, phase shift −2.13, and vertical shift 7.
12. Graph $y = \sin x - \cos x$.
13. **Health** If a person has a blood pressure of 130 over 70, then the person's blood pressure oscillates between the maximum of 130 and a minimum of 70.
    a. Write the equation for the midline about which this person's blood pressure oscillates.
    b. If the person's pulse rate is 60 beats a minute, write a sine equation that models his or her blood pressure using $t$ as time in seconds.
    c. Graph the equation.

## EXERCISES

**Practice**

**State the phase shift for each function. Then graph each function.**

14. $y = \sin(\theta - 2\pi)$
15. $y = \sin(2\theta + \pi)$
16. $y = 2\cos\left(\frac{\theta}{4} + \frac{\pi}{2}\right)$

**State the vertical shift and the equation of the midline for each function. Then graph each function.**

**17.** $y = \sin \frac{\theta}{2} + \frac{1}{2}$ **18.** $y = 5 \cos \theta - 4$ **19.** $y = 7 + \cos 2\theta$

**20.** State the horizontal and vertical shift for $y = -8 \sin (2\theta - 4\pi) - 3$.

**State the amplitude, period, phase shift, and vertical shift for each function. Then graph the function.**

**21.** $y = 3 \cos \left(\theta - \frac{\pi}{2}\right)$

**22.** $y = 6 \sin \left(\theta + \frac{\pi}{3}\right) + 2$

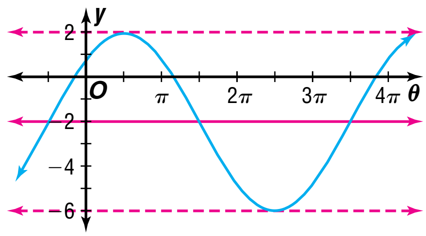
**23.** $y = -2 + \sin \left(\frac{\theta}{3} - \frac{\pi}{12}\right)$

**24.** $y = 20 + 5 \cos (3\theta + \pi)$

**25.** $y = \frac{1}{4} \cos \frac{\theta}{2} - 3$

**26.** $y = 10 \sin \left(\frac{\theta}{4} - 4\pi\right) - 5$

**27.** State the amplitude, period, phase shift, and vertical shift of the sine curve shown at the right.




**Write an equation of the sine function with each amplitude, period, phase shift, and vertical shift.**

**28.** amplitude = 7, period = $3\pi$, phase shift = $\pi$, vertical shift = $-7$

**29.** amplitude = 50, period = $\frac{3\pi}{4}$, phase shift = $\frac{\pi}{2}$, vertical shift = $-25$

**30.** amplitude = $\frac{3}{4}$, period = $\frac{\pi}{5}$, phase shift = $\pi$, vertical shift = $\frac{1}{4}$

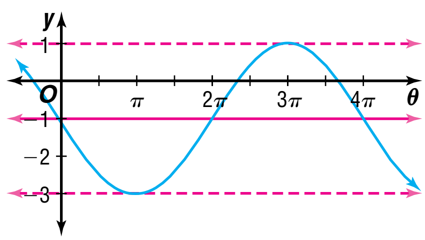
**Write an equation of the cosine function with each amplitude, period, phase shift, and vertical shift.**

**31.** amplitude = 3.5, period = $\frac{\pi}{2}$, phase shift = $\frac{\pi}{4}$, vertical shift = 7

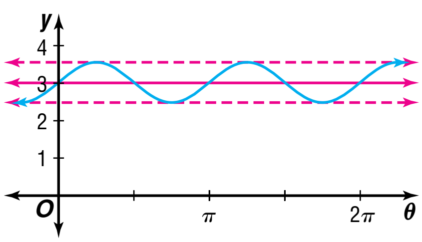
**32.** amplitude = $\frac{4}{5}$, period = $\frac{\pi}{6}$, phase shift = $\frac{\pi}{3}$, vertical shift = $\frac{7}{5}$

**33.** amplitude = 100, period = 45, phase shift = 0, vertical shift = $-110$

**34.** Write a cosine equation for the graph at the right.




**35.** Write a sine equation for the graph at the right.



**Graph each function.**

**36.** $y = \sin x + x$

**37.** $y = \cos x - \sin x$

**38.** $y = \sin x + \sin 2x$

**39.** On the same coordinate plane, graph each function.

**a.** $y = 2 \sin x$ **b.** $y = 3 \cos x$ **c.** $y = 2 \sin x + 3 \cos x$

**40.** Use the graphs of $y = \cos 2x$ and $y = \cos 3x$ to graph $y = \cos 2x - \cos 3x$.

## Applications and Problem Solving

Real World Application

**41. Biology** In the wild, predators such as wolves need prey such as sheep to survive. The population of the wolves and the sheep are cyclic in nature. Suppose the population of the wolves $W$ is modeled by $W = 2000 + 1000 \sin\left(\frac{\pi t}{6}\right)$ and population of the sheep $S$ is modeled by $S = 10{,}000 + 5000 \cos\left(\frac{\pi t}{6}\right)$ where $t$ is the time in months.

**a.** What are the maximum number and the minimum number of wolves?

**b.** What are the maximum number and the minimum number of sheep?

**c.** Use a graphing calculator to graph both equations for values of $t$ from 0 to 24.

**d.** During which months does the wolf population reach a maximum?

**e.** During which months does the sheep population reach a maximum?

**f.** What is the relationship of the maximum population of the wolves and the maximum population of the sheep? Explain.

**42. Critical Thinking** Use the graphs of $y = x$ and $y = \cos x$ to graph $y = x \cos x$.

**43. Entertainment** As you ride a Ferris wheel, the height that you are above the ground varies periodically. Consider the height of the center of the wheel to be the equilibrium point. Suppose the diameter of a Ferris Wheel is 42 feet and travels at a rate of 3 revolutions per minute. At the highest point, a seat on the Ferris wheel is 46 feet above the ground.

**a.** What is the lowest height of a seat?

**b.** What is the equation of the midline?

**c.** What is the period of the function?

**d.** Write a sine equation to model the height of a seat that was at the equilibrium point heading upward when the ride began.

**e.** According to the model, when will the seat reach the highest point for the first time?

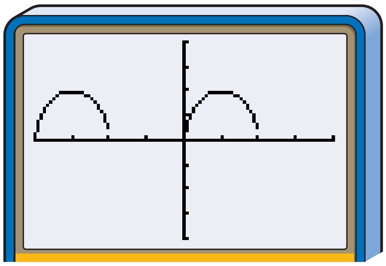
**f.** According to the model, what is the height of the seat after 10 seconds?

**44. Electronics** In electrical circuits, the voltage and current can be described by sine or cosine functions. If the graphs of these functions have the same period, but do not pass through their zero points at the same time, they are said to have a *phase difference.* For example, if the voltage is 0 at 90° and the current is 0 at 180°, they are 90° out of phase. Suppose the voltage across an inductor of a circuit is represented by $y = 2 \cos 2x$ and the current across the component is represented by $y = \cos\left(2x - \frac{\pi}{2}\right)$. What is the phase relationship between the signals?

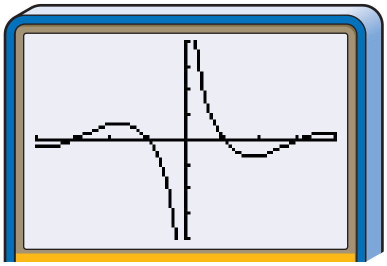
**45. Critical Thinking** The windows for the following calculator screens are set at [$-2\pi$, $2\pi$] scl: $0.5\pi$ by [$-2$, 2] scl: 0.5. Without using a graphing calculator, use the equations below to identify the graph on each calculator screen.

$y = \cos x^2$ $\qquad y = \sqrt{\sin x}$ $\qquad y = \dfrac{\cos x}{x}$ $\qquad y = \sin \sqrt{x}$

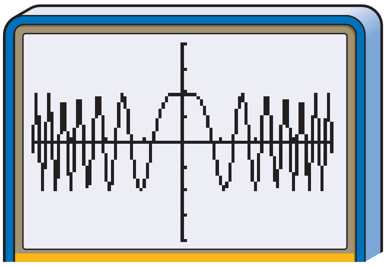
a.



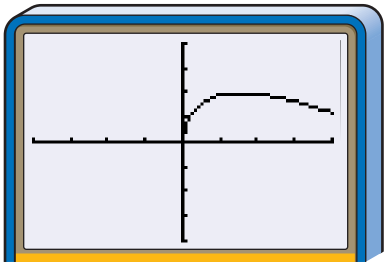
b.



c.



d.



## Mixed Review

**46. Music** Write an equation of the sine function that represents the initial behavior of the vibrations of the note D above middle C having amplitude 0.25 and a frequency of 294 hertz. *(Lesson 6-4)*

**47.** Determine the linear velocity of a point rotating at an angular velocity of 19.2 radians per second at a distance of 7 centimeters from the center of the rotating object. *(Lesson 6-2)*

**48.** Graph $y = \dfrac{x-3}{x-2}$. *(Lesson 3-7)*

**49.** Find the inverse of $f(x) = \dfrac{3}{x-1}$. *(Lesson 3-4)*

**50.** Find matrix $X$ in the equation $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -3 & -5 \end{bmatrix} = X$. *(Lesson 2-3)*

**51.** Solve the system of equations. *(Lesson 2-1)*

$3x + 5y = 4$

$14x - 35y = 21$

**52.** Graph $y \leq |x + 4|$. *(Lesson 1-8)*

**53.** Write the standard form of the equation of the line through the point at $(3, -2)$ that is parallel to the graph of $3x - y + 7 = 0$. *(Lesson 1-5)*

**54. SAT Practice** **Grid-In** A swimming pool is 75 feet long and 42 feet wide. If 7.48 gallons equals 1 cubic foot, how many gallons of water are needed to raise the level of the water 4 inches?



**Extra Practice** See p. A37.