SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

CHAPTER OBJECTIVES

- **Solve systems of equations and inequalities.** *(Lessons 2-1, 2-2, 2-6)*
- **Define matrices.** *(Lesson 2-3)*
- **Add, subtract, and multiply matrices.** *(Lesson 2-3)*
- **Use matrices to model transformations.** *(Lesson 2-4)*
- **Find determinants and inverses of matrices.** *(Lesson 2-5)*
- **Use linear programming to solve problems.** *(Lesson 2-7)*

2-1 Solving Systems of Equations in Two Variables

OBJECTIVES
- Solve systems of equations graphically.
- Solve systems of equations algebraically.

Real World Application

CONSUMER CHOICES Madison is thinking about leasing a car for two years. The dealership says that they will lease her the car she has chosen for \$326 per month with only \$200 down. However, if she pays \$1600 down, the lease payment drops to \$226 per month. What is the break-even point when comparing these lease options? Which 2-year lease should she choose if the down payment is not a problem? *This problem will be solved in Example 4.*

The *break-even point* is the point in time at which Madison has paid the same total amount on each lease. After finding that point, you can more easily determine which of these arrangements would be a better deal. The break-even point can be determined by solving a system of equations.

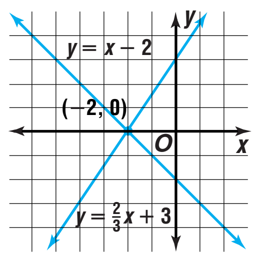
A **system of equations** is a set of two or more equations. To "solve" a system of equations means to find values for the variables in the equations, which make all the equations true at the same time. One way to solve a system of equations is by graphing. The intersection of the graphs represents the point at which the equations have the same x-value and the same y-value. Thus, this ordered pair represents the solution common to both equations. This ordered pair is called the **solution** to the system of equations.

Example 1 **Solve the system of equations by graphing.**

$$\begin{aligned} 3x - 2y &= -6 \\ x + y &= -2 \end{aligned}$$

First rewrite each equation of the system in slope-intercept form by solving for y.

$$\begin{aligned} 3x - 2y &= -6 \\ x + y &= -2 \end{aligned} \quad \text{becomes} \quad \begin{aligned} y &= \frac{3}{2}x + 3 \\ y &= -x - 2 \end{aligned}$$



Since the two lines have different slopes, the graphs of the equations are intersecting lines. The solution to the system is $(-2, 0)$.

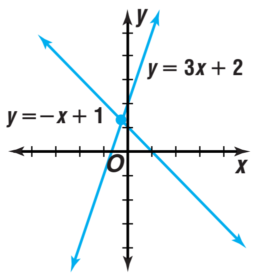
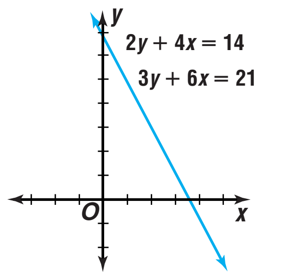
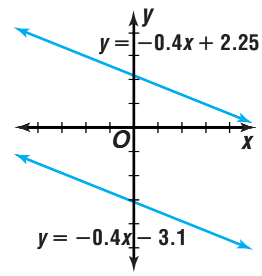
Graphing Calculator Tip

You can estimate the solution to a system of equations by using the **TRACE** function on your graphing calculator.

As you saw in Example 1, when the graphs of two equations intersect there is a solution to the system of equations. However, you may recall that the graphs of two equations may be parallel lines or, in fact, the same line. Each of these situations has a different type of system of linear equations.

A **consistent** system of equations has at least one solution. If there is exactly one solution, the system is **independent.** If there are infinitely many solutions, the system is **dependent.** If there is no solution, the system is **inconsistent.** By rewriting each equation of a system in slope-intercept form, you can more easily determine the type of system you have and what type of solution to expect.

The chart below summarizes the characteristics of these types of systems.

consistent		inconsistent
independent	**dependent**	
		
$y = 3x + 2$ $y = -x + 1$	$2y + 4x = 14 \rightarrow y = -2x + 7$ $3y + 6x = 21 \rightarrow y = -2x + 7$	$y = -0.4x + 2.25$ $y = -0.4x - 3.1$
different slope	same slope, same intercept	same slope, different intercept
Lines intersect.	Graphs are same line.	Lines are parallel.
one solution	infinitely many solutions	no solution

When graphs result in lines that are the same line, we say the lines coincide.

Often, graphing a system of equations is not the best method of finding its solution. This is especially true when the solution to the system contains non-integer values. Systems of linear equations can also be solved algebraically. Two common ways of solving systems algebraically are the **elimination method** and the **substitution method.** In some cases, one method may be easier to use than the other.

Example 2 **Use the elimination method to solve the system of equations.**

$$\mathbf{1.5x + 2y = 20}$$
$$\mathbf{2.5x - 5y = -25}$$

One way to solve this system is to multiply both sides of the first equation by 5, multiply both sides of the second equation by 2, and add the two equations to eliminate y. Then solve the resulting equation.

$$5(1.5x + 2y) = 5(20) \qquad 7.5x + 10y = 100$$
$$2(2.5x - 5y) = 2(-25) \quad \Rightarrow \quad 5x - 10y = -50$$
$$12.5x = 50$$
$$x = 4$$

Now substitute 4 for x in either of the *original* equations.

$$1.5x + 2y = 20$$
$$1.5(4) + 2y = 20 \qquad x = 4$$
$$2y = 14$$
$$y = 7$$

The solution is (4, 7). *Check it by substituting into $2.5x - 5y = -25$. If the coordinates make both equations true, then the solution is correct*

If one of the equations contains a variable with a coefficient of 1, the system can often be solved more easily by using the substitution method.

Example 3 **Use the substitution method to solve the system of equations.**

$$2x + 3y = 8$$
$$x - y = 2$$

You can solve the second equation for either y or x. If you solve for x, the result is $x = y + 2$. Then substitute $y + 2$ for x in the first equation.

$$2x + 3y = 8$$
$$2(y + 2) + 3y = 8 \quad x = y + 2$$
$$5y = 4$$
$$y = \frac{4}{5}$$

Now substitute $\frac{4}{5}$ for y in either of the original equations, and solve for x.

$$x - y = 2$$
$$x - \frac{4}{5} = 2 \quad y = \frac{4}{5}$$
$$x = \frac{14}{5}$$

The solution is $\left(\frac{14}{5}, \frac{4}{5}\right)$.

GRAPHING CALCULATOR EXPLORATION

You can use a graphing calculator to find the solution to an independent system of equations.

- Graph the equations on the same screen.
- Use the **CALC** menu and select **5:intersect** to determine the coordinates of the point of intersection of the two graphs.

TRY THESE

Find the solution to each system.

1. $y = 500x - 20$
$y = -20x + 500$

2. $3x - 4y = 320$
$5x + 2y = 340$

WHAT DO YOU THINK?

3. How accurate are solutions found on the calculator?

4. What type of system do the equations $5x - 7y = 70$ and $10x - 14y = 120$ form? What happens when you try to find the intersection point on the calculator?

5. Graph a system of dependent equations. Find the intersection point. Use the **TRACE** function to move the cursor and find the intersection point again. What pattern do you observe?

You can use a system of equations to solve real-world problems. Choose the best method for solving the system of equations that models the situation.

Example 4 Real World Application

CONSUMER CHOICES **Refer to the application at the beginning of the lesson.**

a. What is the break-even point in the two lease plans that Madison is considering?

b. If Madison keeps the lease for 24 months, which lease should she choose?

a. First, write an equation to represent the amount she will pay with each plan. Let C represent the total cost and m the number of months she has had the lease.

Lease 1 ($200 down with monthly payment of $326): $C = 326m + 200$

Lease 2 ($1600 down with monthly payment of $226): $C = 226m + 1600$

Now, solve the system of equations. Since both equations contain C, we can substitute the value of C from one equation into the other.

(continued on the next page)

$$C = 326m + 200$$

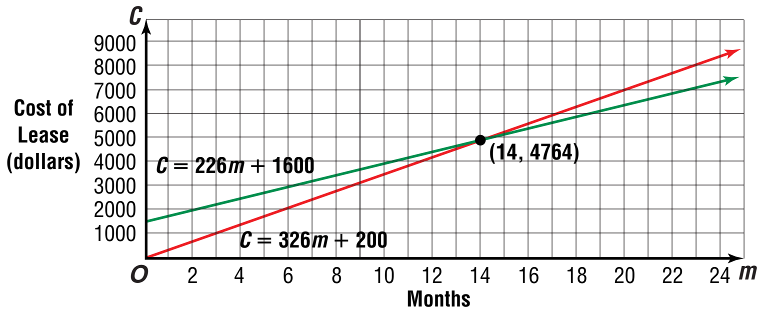
$$226m + 1600 = 326m + 200 \quad C = 226m + 1600$$

$$1400 = 100m$$

$$14 = m$$

With the fourteenth monthly payment, she reaches the break-even point.

b. The graph of the equations shows that after that point, Lease 1 is more expensive for the 2-year lease. So, Madison should probably choose Lease 2.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a system of equations in which it is easier to use the substitution method to solve the system rather than the elimination method. Explain your choice.
2. Refer to the application at the beginning of the lesson. **Explain** what factors Madison might consider before making a decision on which lease to select.
3. *Math Journal* **Write** a description of the three different possibilities that may occur when graphing a system of two linear equations. Include examples and solutions that occur with each possibility.

Guided Practice

4. State whether the system $2y + 3x = 6$ and $4y = 16 - 6x$ is *consistent and independent, consistent and dependent,* or *inconsistent.* Explain your reasoning.

Solve each system of equations by graphing.

5. $y = 5x - 2$
   $y = -2x + 5$

6. $x - y = 2$
   $2x = 2y + 10$

Solve each system of equations algebraically.

7. $7x + y = 9$
   $5x - y = 15$

8. $3x + 4y = -1$
   $6x - 2y = 3$

9. $\frac{1}{3}x - \frac{3}{2}y = -4$
   $5x - 4y = 14$

10. **Sales** HomePride manufactures solid oak racks for displaying baseball equipment and karate belts. They usually sell six times as many baseball racks as karate-belt racks. The net profit is \$3 from each baseball rack and \$5 from each karate-belt rack. If the company wants a total profit of \$46,000, how many of each type of rack should they sell?

www.amc.glencoe.com/self_check_quiz

EXERCISES

Practice

State whether each system is *consistent and independent, consistent and dependent,* or *inconsistent.*

11. $x + 3y = 18$
$-x + 2y = 7$

12. $y = 0.5x$
$2y = x + 4$

13. $-35x + 40y = 55$
$7x = 8y - 11$

Solve each system of equations by graphing.

14. $x = 5$
$4x + 5y = 20$

15. $y = -3$
$2x = 8$

16. $x + y = -2$
$3x - y = 10$

17. $x + 3y = 0$
$2x + 6y = 5$

18. $y = x - 2$
$x - 2y = 4$

19. $3x - 2y = -6$
$x = 12 - 4y$

20. Determine what type of solution you would expect from the system of equations $3x - 8y = 10$ and $16x - 32y = 75$ without graphing the system. Explain how you determined your answer.

Solve each system of equations algebraically.

21. $5x - y = 16$
$2x + 3y = 3$

22. $3x - 5y = -8$
$x + 2y = 1$

23. $y = 6 - x$
$x = 4.5 + y$

24. $2x + 3y = 3$
$12x - 15y = -4$

25. $-3x + 10y = 5$
$2x + 7y = 24$

26. $x = 2y - 8$
$2x - y = -7$

27. $2x + 5y = 4$
$3x + 6y = 5$

28. $\frac{3}{5}x - \frac{1}{6}y = 1$
$\frac{1}{5}x + \frac{5}{6}y = 11$

29. $4x + 5y = -8$
$3x - 7y = 10$

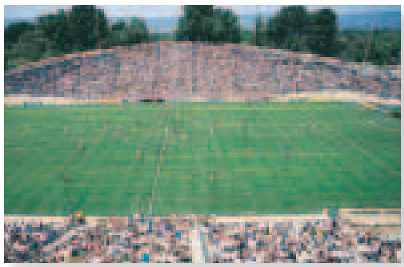
30. Find the solution to the system of equations $3x - y = -9$ and $4x - 2y = -8$.

31. Explain which method seems most efficient to solve the system of equations $a - b = 0$ and $3a + 2b = -15$. Then solve the system.

Applications and Problem Solving

Real World Application

32. **Sports** Spartan Stadium at San Jose State University in California has a seating capacity of about 30,000. A newspaper article states that the Spartans get four times as many tickets as the visiting team. Suppose S represents the number of tickets for the Spartans and V represents the number of tickets for the visiting team's fans.



a. Which system could be used by a newspaper reader to determine how many tickets each team gets?

A $4S + 4V = 30{,}000$
$S = 4V$

B $S - 4V = 0$
$S + V = 30{,}000$

C $S + V = 30{,}000$
$V - 4S = 0$

b. Solve the system to find how many tickets each team gets.

33. **Geometry** Two triangles have the same perimeter of 20 units. One triangle is an isosceles triangle. The other triangle has a side 6 units long. Its other two sides are the same lengths as the base and leg of the isosceles triangle.

a. What are the dimensions of each triangle?

b. What type of triangle is the second triangle?

34. Critical Thinking The solution to a system of two linear equations is $(4, -3)$. One equation has a slope of 4. The slope of the other line is the negative reciprocal of the slope of the first. Find the system of equations.

35. Business The first Earth Day was observed on April 22, 1970. Since then, the week of April 22 has been Earth Week, a time for showing support for environmental causes. Fans Café is offering a reduced refill rate for soft drinks during Earth Week for anyone purchasing a Fans mug. The mug costs $2.95 filled with 16 ounces of soft drink. The refill price is 50¢. A 16-ounce drink in a disposable cup costs $0.85.

a. What is the approximate break-even point for buying the mug and refills in comparison to buying soft drinks in disposable cups?

b. What does this mean? Which offer do you think is best?

c. How would your decision change if the refillable mug offer was extended for a year?

36. Critical Thinking Determine what must be true of *a, b, c, d, e,* and *f* for the system $ax + by = c$ and $dx + ey = f$ to fit each description.

a. consistent and independent

b. consistent and dependent

c. inconsistent

37. Incentive Plans As an incentive plan, a company stated that employees who worked for four years with the company would receive $516 and a laptop computer. Mr. Rodriquez worked for the company for 3.5 years. The company pro-rated the incentive plan, and he still received the laptop computer, but only $264. What was the value of the laptop computer?

38. Ticket Sales In November 1994, the first live concert on the Internet by a major rock'n'roll band was broadcast. Most fans stand in lines for hours to get tickets for concerts. Suppose you are in line for tickets. There are 200 more people ahead of you than behind you in line. The whole line is three times the number of people behind you. How many people are in line for concert tickets?

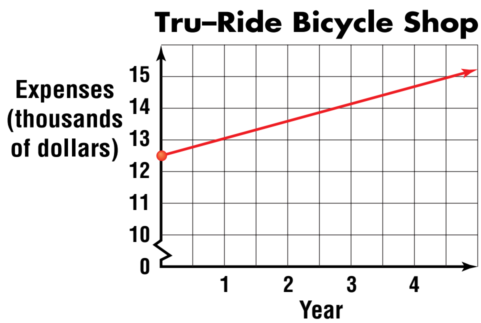
Mixed Review

39. Graph $-2x + 7 \geq y$. *(Lesson 1-8)*

40. Graph $f(x) = 2|x| - 3$. *(Lesson 1-7).*

41. Write an equation of the line parallel to the graph of $y = 2x + 5$ that passes through the point at $(0, 6)$. *(Lesson 1-5)*

42. Manufacturing The graph shows the operational expenses for a bicycle shop during its first four years of business. How much was the startup cost of the business? *(Lesson 1-3)*



43. Find $[f \circ g](x)$ if $f(x) = 3x - 5$ and $g(x) = x + 2$. *(Lesson 1-2)*

44. State the domain and range of the relation $\{(18, -3), (18, 3)\}$. Is this relation a function? Explain. *(Lesson 1-1)*

45. SAT/ACT Practice $\sqrt{\frac{\sqrt{25}}{5}} =$

A 1 **B** $\sqrt{2}$ **C** 2 **D** 5 **E** $5\sqrt{2}$

Extra Practice See p. A28.

2-2 Solving Systems of Equations in Three Variables

OBJECTIVE

- Solve systems of equations involving three variables algebraically.

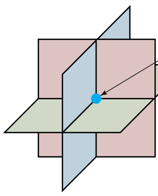
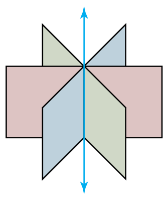
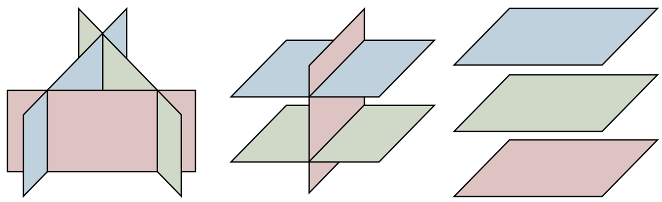
Real World Application

SPORTS In 1998, Cynthia Cooper of the WNBA Houston Comets basketball team was named Team Sportswoman of the Year by the Women's Sports Foundation. Cooper scored 680 points in the 1998 season by hitting 413 of her 1-point, 2-point, and 3-point attempts. She made 40% of her 160 3-point field goal attempts. How many 1-, 2-, and 3-point baskets did Ms. Cooper complete? *This problem will be solved in Example 3.*

You will learn more about graphing in three-dimensional space in Chapter 8.

This situation can be described by a system of three equations in three variables. You used graphing to solve a system of equations in two variables. For a system of equations in three variables, the graph of each equation is a plane in space rather than a line. The three planes can appear in various configurations. This makes solving a system of equations in three variables by graphing rather difficult. However, the pictorial representations provide information about the types of solutions that are possible. Some of them are shown below.

Systems of Equations in Three Variables

Unique Solution	Infinite Solutions	No Solution
		
The three planes intersect at one point.	The three planes intersect in a line.	The three planes have no points in common.

You can solve systems of three equations more efficiently than graphing by using the same algebraic techniques that you used to solve systems of two equations.

Example 1 **Solve the system of equations by elimination.**

$$x - 2y + z = 15$$
$$2x + 3y - 3z = 1$$
$$4x + 10y - 5z = -3$$

One way to solve a system of three equations is to choose pairs of equations and then eliminate one of the variables. Because the coefficient of x is 1 in the first equation, it is a good choice for eliminating x from the second and third equations.

(continued on the next page)

To eliminate x using the first and second equations, multiply each side of the first equation by -2.

$$-2(x - 2y + z) = -2(15)$$
$$-2x + 4y - 2z = -30$$

Then add that result to the second equation.

$$\begin{array}{r} -2x + 4y - 2z = -30 \\ 2x + 3y - 3z = 1 \\ \hline 7y - 5z = -29 \end{array}$$

To eliminate x using the first and third equations, multiply each side of the first equation by -4.

$$-4(x - 2y + z) = -4(15)$$
$$-4x + 8y - 4z = -60$$

Then add that result to the third equation .

$$\begin{array}{r} -4x + 8y - 4z = -60 \\ 4x + 10y - 5z = -3 \\ \hline 18y - 9z = -63 \end{array}$$

Now you have two linear equations in two variables. Solve this system. Eliminate z by multiplying each side of the first equation by -9 and each side of the second equation by 5. Then add the two equations.

$$\begin{array}{r} -9(7y - 5z) = -9(-29) \\ 5(18y - 9z) = 5(-63) \\ \hline \end{array} \Rightarrow \begin{array}{r} -63y + 45z = 261 \\ 90y - 45z = -315 \\ \hline 27y = -54 \\ y = -2 \end{array}$$

The value of y is -2.

By substituting the value of y into one of the equations in two variables, we can solve for the value of z.

$$7y - 5z = -29$$
$$7(-2) - 5z = -29 \quad y = -2$$
$$z = 3 \quad \text{The value of } z \text{ is 3.}$$

Finally, use one of the original equations to find the value of x.

$$x - 2y + z = 15$$
$$x - 2(-2) + 3 = 15 \quad y = -2, z = 3$$
$$x = 8 \quad \text{The value of } x \text{ is 8.}$$

The solution is $x = 8$, $y = -2$, and $z = 3$. This can be written as the **ordered triple** $(8, -2, 3)$. *Check by substituting the values into each of the original equations.*

The substitution method of solving systems of equations also works with systems of three equations.

Example 2 **Solve the system of equations by substitution.**

$$\mathbf{4x = -8z}$$
$$\mathbf{3x - 2y + z = 0}$$
$$\mathbf{-2x + y - z = -1}$$

You can easily solve the first equation for x.

$$4x = -8z$$
$$x = -2z \quad \text{Divide each side by 4.}$$

Then substitute $-2z$ for x in each of the other two equations. Simplify each equation.

$$3x - 2y + z = 0$$
$$3(-2z) - 2y + z = 0 \quad x = -2z$$
$$-2y - 5z = 0$$

$$-2x + y - z = -1$$
$$-2(-2z) + y - z = -1 \quad x = -2z$$
$$y + 3z = -1$$

Solve $y + 3z = -1$ for y.
$$y + 3z = -1$$
$$y = -1 - 3z \quad \textit{Subtract 3z from each side.}$$

Substitute $-1 - 3z$ for y in $-2y - 5z = 0$. Simplify.

$$-2y - 5z = 0$$
$$-2(-1 - 3z) - 5z = 0 \quad y = -1 - 3z$$
$$z = -2$$

Now, find the values of y and x. Use $y = -1 - 3z$ and $x = -2z$. Replace z with -2.

$$y = -1 - 3z$$
$$y = -1 - 3(-2) \quad z = -2$$
$$y = 5$$

$$x = -2z$$
$$x = -2(-2) \quad z = -2$$
$$x = 4$$

The solution is $x = 4$, $y = 5$, *and* $z = -2$. *Check each value in the original system.*

Many real-world situations can be represented by systems of three equations.

Example 3 Real World Application

SPORTS Refer to the application at the beginning of the lesson. Find the number of 1point free throws, 2-point field goals, and 3-point field goals Cynthia Cooper scored in the 1998 season.

Write a system of equations. Define the variables as follows.

x = the number of 1-point free throws

y = the number of 2-point field goals

z = the number of 3-point field goals

The system is:

$$x + 2y + 3z = 680 \quad \textit{total number of points}$$
$$x + y + z = 413 \quad \textit{total number of baskets}$$
$$\frac{z}{160} = 0.40 \quad \textit{percent completion}$$

The third equation is a simple linear equation. Solve for z.

$\frac{z}{160} = 0.40$, so $z = 160(0.40)$ or 64.

Now substitute 64 for z to make a system of two equations.

$$x + 2y + 3z = 680$$
$$x + 2y + 3(64) = 680 \quad z = 64$$
$$x + 2y = 488$$

$$x + y + z = 413$$
$$x + y + 64 = 413 \quad z = 64$$
$$x + y = 349$$

Solve $x + y = 349$ for x. Then substitute that value for x in $x + 2y = 488$ and solve for y.

$$x + y = 349$$
$$x = 349 - y$$

$$x + 2y = 488$$
$$(349 - y) + 2y = 488 \quad x = 349 - y$$
$$y = 139$$

(continued on the next page)

Solve for x.
$x = 349 - y$
$x = 349 - 139 \quad y = 139$
$x = 210$

In 1998, Ms. Cooper made 210 1-point free throws, 139 2-point field goals, and 64 3-point field goals. *Check your answer in the original problem.*

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** solving a system of three equations to solving a system of two equations.
2. **Describe** what you think would happen if two of the three equations in a system were consistent and dependent. Give an example.
3. **Write** an explanation of how to solve a system of three equations using the elimination method.

Guided Practice

Solve each system of equations.

4. $4x + 2y + z = 7$
   $2x + 2y - 4z = -4$
   $x + 3y - 2z = -8$
5. $x - y - z = 7$
   $-x + 2y - 3z = -12$
   $3x - 2y + 7z = 30$
6. $2x - 2y + 3z = 6$
   $2x - 3y + 7z = -1$
   $4x - 3y + 2z = 0$

7. **Physics** The height of an object that is thrown upward with a constant acceleration of a feet per second per second is given by the equation $s = \frac{1}{2}at^2 + v_o t + s_o$. The height is s feet, t represents the time in seconds, v_o is the initial velocity in feet per second, and s_o is the initial height in feet. Find the acceleration, the initial velocity, and the initial height if the height at 1 second is 75 feet, the height at 2.5 seconds is 75 feet, and the height at 4 seconds is 3 feet.

Exercises

Practice

Solve each system of equations.

8. $x + 2y + 3z = 5$
   $3x + 2y - 2z = -13$
   $5x + 3y - z = -11$
9. $7x + 5y + z = 0$
   $-x + 3y + 2z = 16$
   $x - 6y - z = -18$
10. $x - 3z = 7$
    $2x + y - 2z = 11$
    $-x - 2y + 9z = 13$
11. $3x - 5y + z = 9$
    $x - 3y - 2z = -8$
    $5x - 6y + 3z = 15$
12. $8x - z = 4$
    $y + z = 5$
    $11x + y = 15$
13. $4x - 3y + 2z = 12$
    $x + y - z = 3$
    $-2x - 2y + 2z = 5$
14. $36x - 15y + 50z = -10$
    $2x + 25y = 40$
    $54x - 5y + 30z = -160$
15. $-x - 3y + z = 54$
    $4x + 2y - 3z = -32$
    $2y + 8z = 78$
16. $1.8x - z = 0.7$
    $1.2y + z = -0.7$
    $1.5x - 3y = 3$
17. If possible, find the solution of $y = x + 2z$, $z = -1 - 2x$, and $x = y - 14$.
18. What is the solution of $\frac{1}{8}x - \frac{2}{3}y + \frac{5}{6}z = -8$, $\frac{3}{4}x + \frac{1}{6}y - \frac{1}{3}z = -12$, and $\frac{3}{16}x - \frac{5}{8}y - \frac{7}{12}z = -25$? If there is no solution, write *impossible*.

Applications and Problem Solving

Real World Application

19. Finance Ana Colón asks her broker to divide her 401K investment of \$2000 among the International Fund, the Fixed Assets Fund, and company stock. She decides that her investment in the International Fund should be twice her investment in company stock. During the first quarter, the International Fund earns 4.5%, the Fixed Assets Fund earns 2.6%, and the company stock falls 0.2%. At the end of the first quarter, Ms. Colón receives a statement indicating a return of \$58 on her investment. How did the broker divide Ms. Colón's initial investment?

20. Critical Thinking Write a system of equations that fits each description.

a. The system has a solution of $x = -5$, $y = 9$, $z = 11$.

b. There is no solution to the system.

c. The system has an infinite number of solutions.

21. Physics Each year the Punkin' Chunkin' contest is held in Lewes, Delaware. The object of the contest is to propel an 8- to 10-pound pumpkin as far as possible. Steve Young of Hopewell, Illinois, set the 1998 record of 4026.32 feet. Suppose you build a machine that fires the pumpkin so that it is at a height of 124 feet after 1 second, the height at 3 seconds is 272 feet, and the height at 8 seconds is 82 feet. Refer to the formula in Exercise 7 to find the acceleration, the initial velocity, and the initial height of the pumpkin.

22. Critical Thinking Suppose you are using elimination to solve a system of equations.

a. How do you know that a system has no solution?

b. How do you know when it has an infinite number of solutions?

23. Number Theory Find all of the ordered triples (x, y, z) such that when any one of the numbers is added to the product of the other two, the result is 2.

Mixed Review

24. Solve the system of equations, $3x + 4y = 375$ and $5x + 2y = 345$. *(Lesson 2-1)*

25. Graph $y \leq -\frac{1}{3}x + 2$. *(Lesson 1-8)*

26. Show that points with coordinates $(-1, 3)$, $(3, 6)$, $(6, 2)$, and $(2, -1)$ are the vertices of a square. *(Lesson 1-5)*

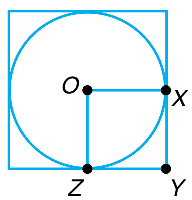
27. Manufacturing It costs ABC Corporation \$3000 to produce 20 of a particular model of color television and \$5000 to produce 60 of that model. *(Lesson 1-4)*

a. Write an equation to represent the cost function.

b. Determine the fixed cost and variable cost per unit.

c. Sketch the graph of the cost function.

28. SAT/ACT Practice In the figure, the area of square $OXYZ$ is 2. What is the area of the circle?



A $\frac{\pi}{4}$ **B** $\pi\sqrt{2}$ **C** 2π

D 4π **E** 8π

Extra Practice See p. A28.

CONTENTS

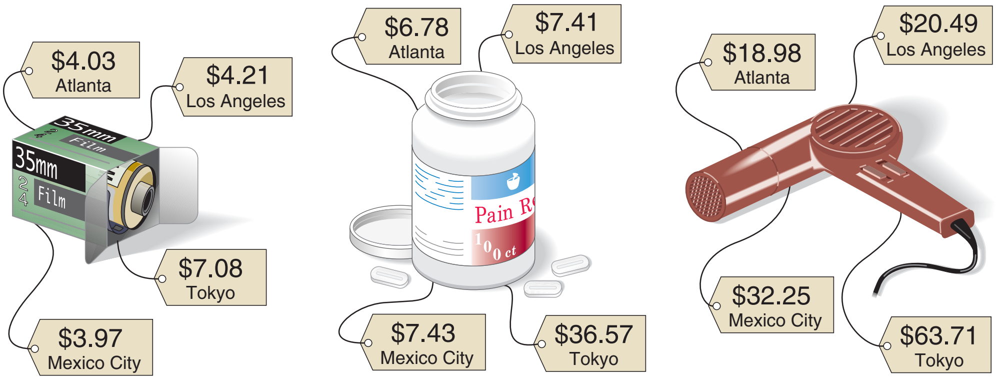
2-3 Modeling Real-World Data with Matrices

OBJECTIVES

- Model data using matrices.
- Add, subtract, and multiply matrices.

Real World Application

TRAVEL Did you ever go on a vacation and realize that you forgot to pack something that you needed? Sometimes purchasing those items while traveling can be expensive. The average cost of some items bought in various cities is given below.



Source: Runzheimer International

Data like these can be displayed or modeled using a matrix. *A problem related to this will be solved in Example 1.*

The plural of matrix is matrices.

A **matrix** is a rectangular array of terms called **elements.** The elements of a matrix are arranged in rows and columns and are usually enclosed by brackets. A matrix with m rows and n columns is an $\boldsymbol{m \times n}$ **matrix** (read "m by n"). The **dimensions** of the matrix are m and n. Matrices can have various dimensions and can contain any type of numbers or other information.

2 × 2 matrix

$$\begin{bmatrix} -\frac{3}{5} & \frac{1}{2} \\ 3 & -\frac{3}{4} \end{bmatrix}$$

2 × 5 matrix

$$\begin{bmatrix} 0.2 & 3.4 & -1.1 & 2.5 & 6.7 \\ 3.4 & -3.4 & -22 & 0.5 & 7.2 \end{bmatrix}$$

3 × 1 matrix

$$\begin{bmatrix} -2 \\ 3 \\ 11 \end{bmatrix}$$

The element 3 is in row 2, column 1.

Special names are given to certain matrices. A matrix that has only one row is called a **row matrix,** and a matrix that has only one column is called a **column matrix.** A **square matrix** has the same number of rows as columns. Sometimes square matrices are called matrices of ***n*th order,** where n is the number of rows and columns. The elements of an $m \times n$ matrix can be represented using double subscript notation; that is, a_{24} would name the element in the second row and fourth column.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} is the element in the ith row and the jth column.

Example 1 **TRAVEL** **Refer to the application at the beginning of the lesson.**

a. Use a matrix to represent the data.

b. Use a symbol to represent the price of pain reliever in Mexico City.

a. To represent data using a matrix, choose which category will be represented by the columns and which will be represented by the rows. Let's use the columns to represent the prices in each city and the rows to represent the prices of each item. Then write each data piece as you would if you were placing the data in a table.

	Atlanta	Los Angeles	Mexico City	Tokyo
film (24 exp.)	\$4.03	\$4.21	\$3.97	\$7.08
pain reliever (100 ct)	\$6.78	\$7.41	\$7.43	\$36.57
blow dryer	\$18.98	\$20.49	\$32.25	\$63.71

Notice that the category names appear outside of the matrix.

b. The price of pain reliever in Mexico City is found in the row 2, column 3 of the matrix. This element is represented by the symbol a_{23}.

Just as with numbers or algebraic expressions, matrices are equal under certain conditions.

Equal Matrices

Two matrices are equal if and only if they have the same dimensions and are identical, element by element.

Example 2 **Find the values of x and y for which the matrix equation $\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 2x - 6 \\ 2y \end{bmatrix}$ is true.**

Since the corresponding elements are equal, we can express the equality of the matrices as two equations.

$y = 2x - 6$
$x = 2y$

Solve the system of equations by using substitution.

$y = 2x - 6$
$y = 2(2y) - 6$ *Substitute 2y for x.*
$y = 2$ *Solve for y.*

$x = 2(2)$ *Substitute 2 for y in the second equation to find x.*
$x = 4$

The matrices are equal if $x = 4$ and $y = 2$. *Check by substituting into the matrices.*

Matrices are usually named using capital letters. The sum of two matrices, $A + B$, exists only if the two matrices have the same dimensions. The ijth element of $A + B$ is $a_{ij} + b_{ij}$.

Addition of Matrices

The sum of two $m \times n$ matrices is an $m \times n$ matrix in which the elements are the sum of the corresponding elements of the given matrices.

Example 3 Find $A + B$ if $A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 5 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 7 & -1 \\ 4 & -3 & 10 \end{bmatrix}$.

$$A + B = \begin{bmatrix} -2 + (-6) & 0 + 7 & 1 + (-1) \\ 0 + 4 & 5 + (-3) & -8 + 10 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 7 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

Graphing Calculator Appendix

For keystroke instruction on entering matrices and performing operations on them, see pages A16-A17.

You know that 0 is the additive identity for real numbers because $a + 0 = a$. Matrices also have additive identities. For every matrix A, another matrix can be found so that their sum is A. For example,

$$\text{if } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called a **zero matrix.** The $m \times n$ zero matrix is the **additive identity matrix** for any $m \times n$ matrix.

You also know that for any number a, there is a number $-a$, called the additive inverse of a, such that $a + (-a) = 0$. Matrices also have additive inverses. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the matrix that must be added to A in order to have a sum of a zero matrix is $\begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}$ or $-A$. Therefore, $-A$ is the additive inverse of A. The additive inverse is used when you subtract matrices.

Subtraction of Matrices

The difference $A - B$ of two $m \times n$ matrices is equal to the sum $A + (-B)$, where $-B$ represents the additive inverse of B.

Example 4 Find $C - D$ if $C = \begin{bmatrix} 9 & 4 \\ -1 & 3 \\ 0 & -4 \end{bmatrix}$ and $D = \begin{bmatrix} 8 & -2 \\ -6 & 1 \\ 5 & -5 \end{bmatrix}$.

$$C - D = C + (-D)$$

$$= \begin{bmatrix} 9 & 4 \\ -1 & 3 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} -8 & 2 \\ 6 & -1 \\ -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 + (-8) & 4 + 2 \\ -1 + 6 & 3 + (-1) \\ 0 + (-5) & -4 + 5 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 6 \\ 5 & 2 \\ -5 & 1 \end{bmatrix}$$

You can multiply a matrix by a number; when you do, the number is called a **scalar.** The product of a scalar and a matrix A is defined as follows.

Scalar Product

The product of a scalar k and an $m \times n$ matrix A is an $m \times n$ matrix denoted by kA. Each element of kA equals k times the corresponding element of A.

Example 5 If $A = \begin{bmatrix} -4 & 1 & -1 \\ 3 & 7 & 0 \\ -3 & -1 & 8 \end{bmatrix}$, find $3A$.

$$3\begin{bmatrix} -4 & 1 & -1 \\ 3 & 7 & 0 \\ -3 & -1 & 8 \end{bmatrix} = \begin{bmatrix} 3(-4) & 3(1) & 3(-1) \\ 3(3) & 3(7) & 3(0) \\ 3(-3) & 3(-1) & 3(8) \end{bmatrix}$$ *Multiply each element by 3.*

$$= \begin{bmatrix} -12 & 3 & -3 \\ 9 & 21 & 0 \\ -9 & -3 & 24 \end{bmatrix}$$

You can also multiply a matrix by a matrix. For matrices A and B, you can find AB if the number of columns in A is the same as the number of rows in B.

$$\begin{bmatrix} 3 & -8 & 1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 0 & 1 \\ -4 & 0 & -2 & 1 \\ 1 & -3 & -1 & 6 \end{bmatrix}$$

2 × 3 *3 × 4*

Since 3 = 3, multiplication is possible.

$$\begin{bmatrix} 5 & 3 & 1 & 0 \\ 6 & 0 & 2 & -3 \\ -5 & 3 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 12 & 9 \\ 0 & 0 & -4 & -8 \\ -2 & 3 & 4 & 3 \end{bmatrix}$$

3 × 4 *3 × 4*

Since 4 ≠ 3, multiplication is not possible.

The product of two matrices is found by multiplying columns and rows. Suppose $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$. Each element of matrix AX is the product of one row of matrix A and one column of matrix X.

$$AsX = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_2 + b_2y_2 \end{bmatrix}$$

In general, the product of two matrices is defined as follows.

Product of Two Matrices

The product of an $m \times n$ matrix A and an $n \times r$ matrix B is an $m \times r$ matrix AB. The ijth element of AB is the sum of the products of the corresponding elements in the ith row of A and the jth column of B.

Example 6 **Use matrices $A = \begin{bmatrix} 7 & 0 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -3 & 6 \\ 5 & 4 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 6 & -1 & 4 \\ 2 & -2 & -1 \end{bmatrix}$ to find each product.**

a. ***AB***

$$AB = \begin{bmatrix} 7 & 0 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -3 & 6 \\ 5 & 4 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7(3) + 0(5) & 7(-3) + 0(4) & 7(6) + 0(-2) \\ 5(3) + 3(5) & 5(-3) + 3(4) & 5(6) + 3(-2) \end{bmatrix} \text{ or } \begin{bmatrix} 21 & -21 & 42 \\ 30 & -3 & 24 \end{bmatrix}$$

b. ***BC***

B is a 2×3 matrix and C is a 2×3 matrix. Since B does not have the same number of columns as C has rows, the product BC does not exist. BC is undefined.

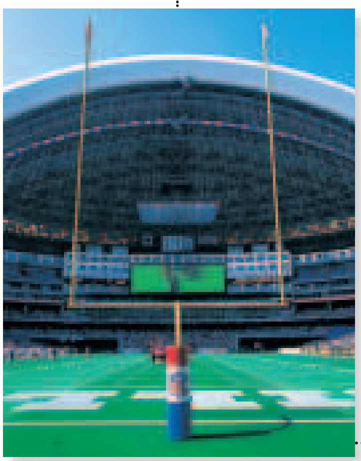
Example 7 **SPORTS** **In football, a player scores 6 points for a touchdown (TD), 3 points for kicking a field goal (FG), and 1 point for kicking the extra point after a touchdown (PAT). The chart lists the records of the top five all-time professional football scorers (as of the end of the 1997 season). Use matrix multiplication to find the number of points each player scored.**

Real World Application

Scorer	TD	FG	PAT
George Blanda	9	335	943
Nick Lowery	0	383	562
Jan Stenerud	0	373	580
Gary Anderson	0	385	526
Morten Andersen	0	378	507

Source: *The World Almanac and Book of Facts*, 1999

Write the scorer information as a 5×3 matrix and the points per play as a 3×1 matrix. Then multiply the matrices.



$$\begin{array}{l} \text{Blanda} \\ \text{Lowery} \\ \text{Stenerud} \\ \text{Anderson} \\ \text{Andersen} \end{array} \overset{\begin{array}{ccc}\text{TD} & \text{FG} & \text{PAT}\end{array}}{\begin{bmatrix} 9 & 335 & 943 \\ 0 & 383 & 562 \\ 0 & 373 & 580 \\ 0 & 385 & 526 \\ 0 & 378 & 507 \end{bmatrix}} \cdot \begin{array}{l} \text{TD} \\ \text{FG} \\ \text{PAT} \end{array} \overset{\text{pts}}{\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}} =$$

$$\begin{array}{l} \text{Blanda} \\ \text{Lowery} \\ \text{Stenerud} \\ \text{Anderson} \\ \text{Andersen} \end{array} \overset{\text{pts}}{\begin{bmatrix} 9(6) + 335(3) + 943(1) \\ 0(6) + 383(3) + 562(1) \\ 0(6) + 373(3) + 580(1) \\ 0(6) + 385(3) + 526(1) \\ 0(6) + 378(3) + 507(1) \end{bmatrix}} = \begin{array}{l} \text{Blanda} \\ \text{Lowery} \\ \text{Stenerud} \\ \text{Anderson} \\ \text{Andersen} \end{array} \overset{\text{pts}}{\begin{bmatrix} 2002 \\ 1711 \\ 1699 \\ 1681 \\ 1641 \end{bmatrix}}$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a matrix other than the one given in Example 1 to represent the data on travel prices.
2. **Tell** the dimensions of the matrix $\begin{bmatrix} 4 & 0 & -2 & 4 \\ -1 & 3 & -1 & 5 \end{bmatrix}$.
3. **Explain** how you determine whether the sum of two matrices exists.
4. **You Decide** Sarah says that $\begin{bmatrix} 3 & 2 & 3 \\ -4 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ is a third-order matrix. Anthony disagrees. Who is correct and why?

Guided Practice

Find the values of *x* and *y* for which each matrix equation is true.

5. $\begin{bmatrix} 2y \\ x \end{bmatrix} = \begin{bmatrix} x - 3 \\ y + 5 \end{bmatrix}$
6. $[18 \quad 24] = [4x - y \quad 12y]$
7. $[16 \quad 0 \quad 2x] = [4x \quad y \quad 8 - y]$

Use matrices *X*, *Y*, and *Z* to find each of the following. If the matrix does not exist, write *impossible*.

$X = \begin{bmatrix} 4 & 1 \\ -2 & 6 \end{bmatrix}$ $\quad Y = \begin{bmatrix} 0 & -3 \end{bmatrix}$ $\quad Z = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

8. $X + Z$ **9.** $Z - Y$ **10.** $Z - X$ **11.** $4X$ **12.** ZY **13.** YX

14. Advertising A newspaper surveyed companies on the annual amount of money spent on television commercials and the estimated number of people who remember seeing those commercials each week. A soft-drink manufacturer spends $40.1 million a year and estimates 78.6 million people remember the commercials. For a package-delivery service, the budget is $22.9 million for 21.9 million people. A telecommunications company reaches 88.9 million people by spending a whopping $154.9 million. Use a matrix to represent this data.

EXERCISES

Practice

Find the values of *x* and *y* for which each matrix equation is true.

15. $\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 2x - 1 \\ y - 5 \end{bmatrix}$

16. $\begin{bmatrix} 9 & 13 \end{bmatrix} = \begin{bmatrix} x + 2y & 4x + 1 \end{bmatrix}$

17. $\begin{bmatrix} 4x \\ 5 \end{bmatrix} = \begin{bmatrix} 15 + x \\ 2y \end{bmatrix}$

18. $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 2y & 2x - 6 \end{bmatrix}$

19. $\begin{bmatrix} 27 \\ 8 \end{bmatrix} = \begin{bmatrix} 3y \\ 5x - 3y \end{bmatrix}$

20. $\begin{bmatrix} 4x - 3y \\ x + y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

21. $\begin{bmatrix} 2x & y & -y \end{bmatrix} = \begin{bmatrix} -10 & 3x & 15 \end{bmatrix}$

22. $\begin{bmatrix} -12 \\ 2 \\ 12y \end{bmatrix} = \begin{bmatrix} 6x \\ y + 1 \\ 10 - x \end{bmatrix}$

23. $\begin{bmatrix} x + y & 3 \\ y & 6 \end{bmatrix} = \begin{bmatrix} 0 & 2y - x \\ y^2 & 4 - 2x \end{bmatrix}$

24. $\begin{bmatrix} x^2 + 1 & 5 - y \\ x + y & y - 4 \end{bmatrix} = \begin{bmatrix} 2 & x \\ 5 & 2 \end{bmatrix}$

25. Find the values of x, y, and z for $3\begin{bmatrix} x & y - 1 \\ 4 & 3z \end{bmatrix} = \begin{bmatrix} 15 & 6 \\ 6z & 3x + y \end{bmatrix}$.

26. Solve $-2\begin{bmatrix} w + 5 & x - z \\ 3y & 8 \end{bmatrix} = \begin{bmatrix} -16 & -4 \\ 6 & 2x + 8z \end{bmatrix}$ for w, x, y, and z.

Use matrices *A*, *B*, *C*, *D*, *E*, and *F* to find each of the following. If the matrix does not exist, write *impossible*.

$A = \begin{bmatrix} 5 & 7 \\ -6 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 5 \\ -1 & 8 \end{bmatrix}$ $\quad C = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 0 & -1 \\ 9 & 0 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 0 \\ 4 & 4 & -2 \end{bmatrix}$

$E = \begin{bmatrix} 8 & -4 & 2 \\ 3 & 1 & -5 \end{bmatrix}$ $\quad F = \begin{bmatrix} -6 & -1 & 0 \\ 1 & 4 & 0 \end{bmatrix}$

27. $A + B$ **28.** $A + C$ **29.** $D + B$ **30.** $D + C$ **31.** $B - A$

32. $C - D$ **33.** $4D$ **34.** $-2F$ **35.** $F - E$ **36.** $E - F$

37. $5A$ **38.** BA **39.** CF **40.** FC **41.** ED

42. AA **43.** $E + FD$ **44.** $-3AB$ **45.** $(BA)E$ **46.** $F - 2EC$

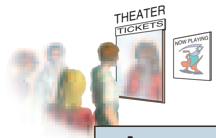
47. Find $3XY$ if $X = \begin{bmatrix} 2 & 4 \\ 8 & -4 \\ -2 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -3 & 6 \\ 5 & 4 & -2 \end{bmatrix}$.

48. If $K = \begin{bmatrix} 1 & -7 \\ 3 & 2 \end{bmatrix}$ and $J = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}$, find $2K - 3J$.

Applications and Problem Solving

Real World Application

49. Entertainment How often do you go to the movies? The graph below shows the projected number of adults of different ages who attend movies at least once a month. Organize the information in a matrix.



Projected Movie Attendance

Age group	Year 1996	2000	2006
18 to 24	8485	8526	8695
25 to 34	10,102	9316	9078
35 to 44	8766	9039	8433
45 to 54	6045	6921	7900
55 to 64	2444	2741	3521
65 and older	2381	2440	2572

Source: American Demographics

interNET CONNECTION

Data Update For the latest National Endowment for the Arts survey, visit **www.amc.glencoe.com**

50. Music The National Endowment for the Arts exists to broaden public access to the arts. In 1992, it performed a study to find what types of arts were most popular and how they could attract more people. The matrices below represent their findings.

Percent of People Listening or Watching Performances

1982

$$\begin{array}{r} \\ \text{Classical} \\ \text{Jazz} \\ \text{Opera} \\ \text{Musicals} \end{array} \begin{array}{c} \begin{array}{ccc} \text{TV} & \text{Radio} & \text{Recording} \end{array} \\ \begin{bmatrix} 25 & 20 & 22 \\ 18 & 18 & 20 \\ 12 & 7 & 8 \\ 21 & 4 & 8 \end{bmatrix} \end{array}$$

1992

$$\begin{array}{r} \\ \text{Classical} \\ \text{Jazz} \\ \text{Opera} \\ \text{Musicals} \end{array} \begin{array}{c} \begin{array}{ccc} \text{TV} & \text{Radio} & \text{Recording} \end{array} \\ \begin{bmatrix} 25 & 31 & 24 \\ 21 & 28 & 21 \\ 12 & 9 & 7 \\ 15 & 4 & 6 \end{bmatrix} \end{array}$$

Source: National Endowment for the Arts

a. Find the difference in arts patronage from 1982 to 1992. Express your answer as a matrix.

b. Which areas saw the greatest increase and decrease in this time?

51. Critical Thinking Consider the matrix equation $\begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix}$.

a. Find the values of a, b, c, and d to make the statement true.

b. If any matrix containing two columns were multiplied by the matrix containing a, b, c, and d, what would you expect the result to be? Explain.

52. **Finance** Investors choose different stocks to comprise a balanced portfolio. The matrix below shows the prices of one share of each of several stocks on the first business day of July, August, and September of 1998.

	July	August	September
Stock A	$\$33\frac{13}{16}$	$\$30\frac{15}{16}$	$\$27\frac{1}{4}$
Stock B	$\$15\frac{1}{16}$	$\$13\frac{1}{4}$	$\$8\frac{3}{4}$
Stock C	$54	$54	$\$46\frac{7}{16}$
Stock D	$\$52\frac{1}{16}$	$\$44\frac{11}{16}$	$\$34\frac{3}{8}$

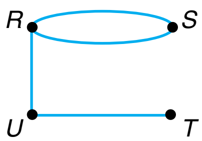
a. Mrs. Numkena owns 42 shares of stock A, 59 shares of stock B, 21 shares of stock C, and 18 shares of stock D. Write a row matrix to represent Mrs. Numkena's portfolio.

b. Use matrix multiplication to find the total value of Mrs. Numkena's portfolio for each month to the nearest cent.

53. **Critical Thinking** Study the matrix at the right. In which row and column will 2001 occur? Explain your reasoning.

$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & 20 & \cdots \\ 4 & 8 & 13 & 19 & 26 & \cdots \\ 7 & 12 & 18 & 25 & 33 & \cdots \\ 11 & 17 & 24 & 32 & 41 & \cdots \\ 16 & 23 & 31 & 40 & 50 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

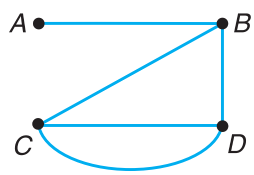
54. **Discrete Math** Airlines and other businesses often use *finite graphs* to represent their routes. A finite graph contains points called *nodes* and segments called *edges*. In a graph for an airline, each node represents a city, and each edge represents a route between the cities. Graphs can be represented by square matrices. The elements of the matrix are the numbers of edges between each pair of nodes. Study the graph and its matrix at the right.



$$\begin{array}{c} \\ R \\ S \\ T \\ U \end{array} \begin{array}{c} \begin{matrix} R & S & T & U \end{matrix} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

a. Represent the graph with nodes *A*, *B*, *C*, and *D* at the right using a matrix.

b. Equivalent graphs have the same number of nodes and edges between corresponding nodes. Can different graphs be represented by the same matrix? Explain your answer.



Mixed Review

55. Solve the system $2x + 6y + 8z = 5$, $-2x + 9y + 12z = 5$, and $4x + 6y - 4z = 3$. *(Lesson 2-2)*

56. State whether the system $4x - 2y = 7$ and $-12x + 6y = -21$ is *consistent and independent, consistent and dependent,* or *inconsistent.* *(Lesson 2-1)*

57. Graph $-6 \leq 3x - y \leq 12$. *(Lesson 1-8)*

58. Graph $f(x) = |3x| + 2$. *(Lesson 1-7)*

59. **Education** Many educators believe that taking practice tests will help students succeed on actual tests. The table below shows data gathered about students studying for an algebra test. Use the data to write a prediction equation. *(Lesson 1-6)*

Practice Test Time (minutes)	15	75	60	45	90	60	30	120	10	120
Test scores (percents)	68	87	92	73	95	83	77	98	65	94

60. Write the slope-intercept form of the equation of the line through points at (1, 4) and (5, 7). *(Lesson 1-4)*

61. Find the zero of $f(x) = 5x - 3$ *(Lesson 1-3)*

62. Find $[f \cdot g](x)$ if $f(x) = \frac{2}{5}x$ and $g(x) = 40x - 10$. *(Lesson 1-2)*

63. Given $f(x) = 4 + 6x - x^3$, find $f(14)$. *(Lesson 1-1)*

64. **SAT/ACT Practice** If $\frac{2x - 3}{x} = \frac{3 - x}{2}$, which of the following could be a value of x?

A -3 **B** -1 **C** 37 **D** 5 **E** 15

GRAPHING CALCULATOR EXPLORATION

Remember the properties of real numbers:

Properties of Addition

Commutative $a + b = b + a$

Associative $(a + b) + c = a + (b + c)$

Properties of Multiplication

Commutative $ab = ba$

Associative $(ab)c = a(bc)$

Distributive Property

$a(b + c) = ab + ac$

Do these properties hold for operations with matrices?

TRY THESE

Use matrices *A*, *B*, and *C* to investigate each of the properties shown above.

$A = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 \\ -2 & -3 \end{bmatrix}$ $C = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}$

WHAT DO YOU THINK?

1. Do these three matrices satisfy the properties of real numbers listed at the left? Explain.

2. Would these properties hold for any 2×2 matrices? Prove or disprove each statement below using variables as elements of each 2×2 matrix.

a. Addition of matrices is commutative.

b. Addition of matrices is associative.

c. Multiplication of matrices is commutative.

d. Multiplication of matrices is associative.

3. Which properties do you think hold for $n \times n$ matrices? Explain.

Extra Practice See p. A28.

CONTENTS

GRAPHING CALCULATOR EXPLORATION

2-4A Transformation Matrices

OBJECTIVE

- Determine the effect of matrix multiplication on a vertex matrix.

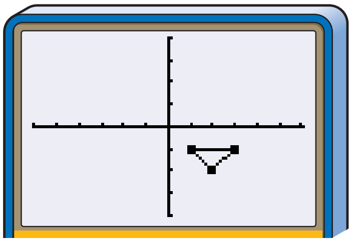
An Introduction to Lesson 2-4

The coordinates of a figure can be represented by a matrix with the x-coordinates in the first row and the y-coordinates in the second. When this matrix is multiplied by certain other matrices, the result is a new matrix that contains the coordinates of the vertices of a different figure. You can use **List** and **Matrix** operations on your calculator to visualize some of these multiplications.

TRY THESE

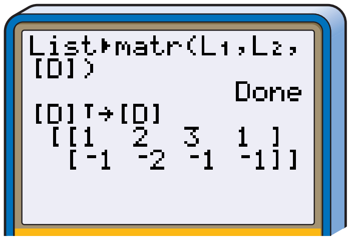
Step 1 Enter $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Step 2 Graph the triangle LMN with $L(1, -1)$, $M(2, -2)$, and $N(3, -1)$ by using **STAT PLOT**. Enter the x-coordinates in **L1** and the y-coordinates in **L2**, repeating the first point to close the figure. In **STAT PLOT**, turn **Plot 1** on and select the connected graph. After graphing the figure, press ZOOM **5** to reflect a less distorted viewing window.



[−4.548 …, 4.548 …] scl: 1 by [−3, 3] scl: 1

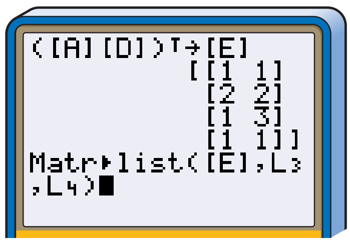
Step 3 To transfer the coordinates from the lists to a matrix, use the **9:List▶matr** command from the **MATH** submenu in the **MATRX** menu. Store this as matrix D. Matrix D has the x-coordinates in Column 1 and y-coordinates in Column 2. But a vertex matrix has the x-coordinates in *Row* 1 and the y-coordinates in *Row 2*. This switch can be easily done by using the **2:ᵀ** (transpose) command found in the **MATH** submenu as shown in the screen.



The **List▶matr** *and* **Matr▶list** *commands transfer the data column for column. That is, the data in List 1 goes to Column 1 of the matrix and vice versa.*

Step 4 Multiply matrix D by matrix A. To graph the result we need to put the ordered pairs back into the **LIST** menu.

- This means we need to transpose AD first. Store as new matrix E.
- Use the **8:Matr▶list** command from the math menu to store values into **L3** and **L4**.



Step 5 Assign **Plot 2** as a connected graph of the **L3** and **L4** data and view the graph.

WHAT DO YOU THINK?

1. What is the relationship between the two plots?
2. Repeat Steps 4 and 5 replacing matrix A with matrix B. Compare the graphs.
3. Repeat Steps 4 and 5 replacing matrix A with matrix C. Compare the graphs.
4. Select a new figure and repeat this activity using each of the 2×2 matrices. Make a conjecture about these 2×2 matrices.

2-4 Modeling Motion with Matrices

OBJECTIVE

- Use matrices to determine the coordinates of polygons under a given transformation.

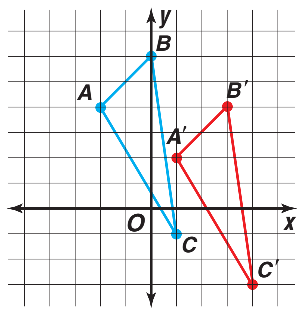
Real World Application

COMPUTER ANIMATION In 1995, animation took a giant step forward with the release of the first major motion picture to be created entirely on computers. Animators use computer software to create three-dimensional computer models of characters, props, and sets. These computer models describe the shape of the object as well as the motion controls that the animators use to create movement and expressions. The animation models are actually very large matrices.

Even though large matrices are used for computer animation, you can use a simple matrix to describe many of the motions called **transformations** that you learned about in geometry. Some of the transformations we will examine in this lesson are **translations** (slides), **reflections** (flips), **rotations** (turns), and **dilations** (enlargements or reductions).

An n-gon is a polygon with n sides.

A $2 \times n$ matrix can be used to express the vertices of an n-gon with the first row of elements representing the x-coordinates and the second row the y-coordinates of the vertices.



Triangle ABC can be represented by the following **vertex matrix.**

$$\begin{array}{r} \\ x\text{-coordinate} \\ y\text{-coordinate} \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} -2 & 0 & 1 \\ 4 & 6 & -1 \end{bmatrix} \end{array}$$

Triangle $A'B'C'$ is congruent to and has the same orientation as $\triangle ABC$, but is moved 3 units right and 2 units down from $\triangle ABC$'s location. The coordinates of $\triangle A'B'C'$ can be expressed as the following vertex matrix.

$$\begin{array}{r} \\ x\text{-coordinate} \\ y\text{-coordinate} \end{array} \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & -3 \end{bmatrix} \end{array}$$

Compare the two matrices. If you add $\begin{bmatrix} 3 & 3 & 3 \\ -2 & -2 & -2 \end{bmatrix}$ to the first matrix, you get the second matrix. Each 3 represents moving 3 units right for each x-coordinate. Likewise, each -2 represents moving 2 units down for each y-coordinate. This type of matrix is called a **translation matrix.** In this transformation, $\triangle ABC$ is the **pre-image,** and $\triangle A'B'C'$ is the **image** after the translation.

Example 1 **Suppose quadrilateral *ABCD* with vertices $A(-1, 1)$, $B(4, 0)$, $C(4, -5)$, and $D(-1, -3)$ is translated 2 units left and 4 units up.**

Note that the image under a translation is the same shape and size as the pre-image. The figures are congruent.

a. Represent the vertices of the quadrilateral as a matrix.

b. Write the translation matrix.

c. Use the translation matrix to find the vertices of $A'B'C'D'$, the translated image of the quadrilateral.

d. Graph quadrilateral *ABCD* and its image.

a. The matrix representing the coordinates of the vertices of quadrilateral *ABCD* will be a 2×4 matrix.

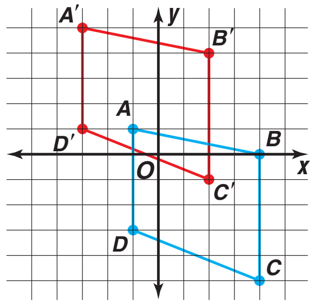
$$\begin{matrix} \\ x\text{-coordinate} \\ y\text{-coordinate} \end{matrix} \begin{matrix} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 0 & -5 & -3 \end{bmatrix} \end{matrix}$$

b. The translation matrix is $\begin{bmatrix} -2 & -2 & -2 & -2 \\ 4 & 4 & 4 & 4 \end{bmatrix}$.

c. Add the two matrices.

$$\begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 0 & -5 & -3 \end{bmatrix} + \begin{bmatrix} -2 & -2 & -2 & -2 \\ 4 & 4 & 4 & 4 \end{bmatrix} = \begin{matrix} \begin{matrix} A' & B' & C' & D' \end{matrix} \\ \begin{bmatrix} -3 & 2 & 2 & -3 \\ 5 & 4 & -1 & 1 \end{bmatrix} \end{matrix}$$

d. Graph the points represented by the resulting matrix.

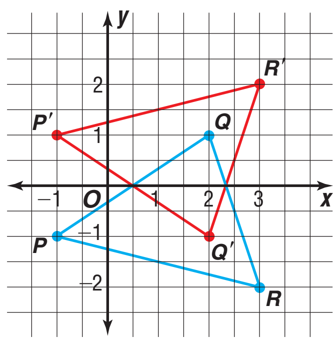


There are three lines over which figures are commonly reflected.

- the x-axis
- the y-axis, and
- the line $y = x$

The preimage and the image under a reflection are congruent.

In the figure at the right, $\triangle P'Q'R'$ is a reflection of $\triangle PQR$ over the x-axis. There is a 2×2 **reflection matrix** that, when multiplied by the vertex matrix of $\triangle PQR$, will yield the vertex matrix of $\triangle P'Q'R'$.



Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represent the unknown square matrix.

Thus, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 3 \\ -1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix}$, or

$\begin{bmatrix} -a - b & 2a + b & 3a - 2b \\ -c - d & 2c + d & 3c - 2d \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix}$.

Since corresponding elements of equal matrices are equal, we can write equations to find the values of the variables. These equations form two systems.

$$-a - b = -1 \qquad -c - d = 1$$
$$2a + b = 2 \qquad 2c + d = -1$$
$$3a - 2b = 3 \qquad 3c - 2d = 2$$

When you solve each system of equations, you will find that $a = 1$, $b = 0$, $c = 0$, and $d = -1$. Thus, the matrix that results in a reflection over the x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This matrix will work for any reflection over the x-axis.

The matrices for a reflection over the y-axis or the line $y = x$ can be found in a similar manner. These are summarized below.

Reflection Matrices		
For a reflection over the:	**Symbolized by:**	**Multiply the vertex matrix by:**
x-axis	$R_{x\text{-axis}}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
y-axis	$R_{y\text{-axis}}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
line $y = x$	$R_{y = x}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 2 Real World Application

ANIMATION **To create an image that appears to be reflected in a mirror, an animator will use a matrix to reflect an image over the y-axis. Use a reflection matrix to find the coordinates of the vertices of a star reflected in a mirror (the y-axis) if the coordinates of the points connected to create the star are (−2, 4), (−3.5, 4), (−4, 5), (−4.5, 4), (−6, 4), (−5, 3), (−5, 1), (−4, 2), (−3, 1), and (−3, 3).**

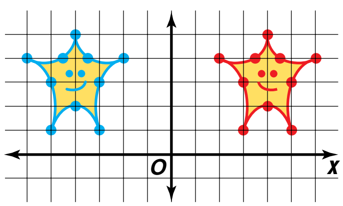
First write the vertex matrix for the points used to define the star.

$$\begin{bmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix}$$

Multiply by the y-axis reflection matrix.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -3.5 & -4 & -4.5 & -6 & -5 & -5 & -4 & -3 & -3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix} =$$
$$\begin{bmatrix} 2 & 3.5 & 4 & 4.5 & 6 & 5 & 5 & 4 & 3 & 3 \\ 4 & 4 & 5 & 4 & 4 & 3 & 1 & 2 & 1 & 3 \end{bmatrix}$$

The vertices used to define the reflection are (2, 4), (3.5, 4), (4, 5), (4.5, 4), (6, 4), (5, 3), (5, 1), (4, 2), (3, 1), and (3, 3).



The preimage and the image under a rotation are congruent.

You may remember from geometry that a rotation of a figure on a coordinate plane can be achieved by a combination of reflections. For example, a 90° counterclockwise rotation can be found by first reflecting the image over the x-axis and then reflecting the reflected image over the line $y = x$. The **rotation matrix,** Rot_{90}, can be found by a composition of reflections. Since reflection matrices are applied using multiplication, the composition of two reflection matrices is a product. Remember that $[f \circ g](x)$ means that you find $g(x)$ first and then evaluate the result for $f(x)$. So, to define Rot_{90}, we use

$$Rot_{90} = R_{y=x} \circ R_{x\text{-axis}} \text{ or } Rot_{90} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Similarly, a rotation of 180° would be rotations of 90° twice or $Rot_{90} \circ Rot_{90}$. A rotation of 270° is a composite of Rot_{180} and Rot_{90}. The results of these composites are shown below.

Rotation Matrices		
For a counterclockwise rotation about the origin of:	**Symbolized by:**	**Multiply the vertex matrix by:**
90°	Rot_{90}	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
180°	Rot_{180}	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
270°	Rot_{270}	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Example 3 Real World Application

ANIMATION **Suppose a figure is animated to spin around a certain point. Numerous rotation images would be necessary to make a smooth movement image. If the image has key points at (1, 1), (−1, 4), (−2, 4), and, (−2, 3) and the rotation is about the origin, find the location of these points at the 90°, 180°, and 270° counterclockwise rotations.**

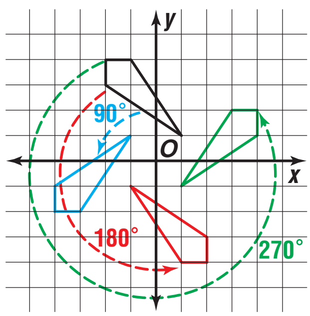
First write the vertex matrix. Then multiply it by each rotation matrix.

The vertex matrix is $\begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix}$.

$$Rot_{90} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -4 & -3 \\ 1 & -1 & -2 & -2 \end{bmatrix}$$

$$Rot_{180} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 2 \\ -1 & -4 & -4 & -3 \end{bmatrix}$$

$$Rot_{180} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -2 & -2 \\ 1 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 & 3 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$



All of the transformations we have discussed have maintained the shape and size of the figure. However, a dilation changes the size of the figure. The dilated figure is similar to the original figure. Dilations using the origin as a center of projection can be achieved by multiplying the vertex matrix by the scale factor needed for the dilation. *All dilations in this lesson are with respect to the origin.*

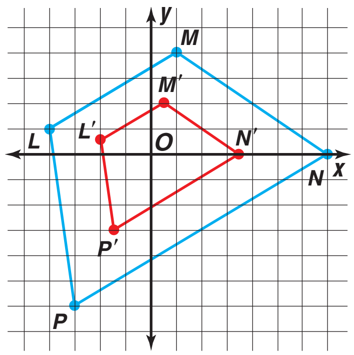
Example 4 **A trapezoid has vertices at $L(-4, 1)$, $M(1, 4)$, $N(7, 0)$, and $P(-3, -6)$. Find the coordinates of the dilated trapezoid $L'M'N'P'$ for a scale factor of 0.5. Describe the dilation.**

First write the coordinates of the vertices as a matrix. Then do a scalar multiplication using the scale factor.

$$0.5\begin{bmatrix} -4 & 1 & 7 & -3 \\ 1 & 4 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 0.5 & 3.5 & -1.5 \\ 0.5 & 2 & 0 & -3 \end{bmatrix}$$

The vertices of the image are $L'(-2, 0.5)$, $M'(0.5, 2)$, $N'(3.5, 0)$, and $P'(-1.5, -3)$.

The image has sides that are half the length of the original figure.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Name** all the transformations described in this lesson. Tell how the pre-image and image are related in each type of transformation.

2. **Explain** how 90°, 180°, and 270° counterclockwise rotations correspond to clockwise rotations.

3. *Math Journal* **Describe** a way that you can remember the elements of the reflection matrices if you forget where the 1s, −1s, and 0s belong.

4. **Match** each matrix with the phrase that best describes its type.

a. $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(1) dilation of scale factor 2

(2) reflection over the y-axis

(3) reflection over the line $y = x$

(4) rotation of 90° counterclockwise about the origin

(5) rotation of 180° about the origin

(6) translation 1 unit left and 1 unit up

Guided Practice **Use matrices to perform each transformation. Then graph the pre-image and the image on the same coordinate grid.**

5. Triangle JKL has vertices $J(-2, 5)$, $K(1, 3)$, and $L(0, -2)$. Use scalar multiplication to find the coordinates of the triangle after a dilation of scale factor 1.5.

6. Square $ABCD$ has vertices $A(-1, 3)$, $B(3, 3)$, $C(3, -1)$, and $D(-1, -1)$. Find the coordinates of the square after a translation of 1 unit left and 2 units down.

7. Square $ABCD$ has vertices at $(-1, 2)$, $(-4, 1)$, $(-3, -2)$, and $(0, -1)$. Find the image of the square after a reflection over the y-axis.

8. Triangle PQR is represented by the matrix $\begin{bmatrix} 3 & -1 & 1 \\ 2 & 4 & -2 \end{bmatrix}$. Find the image of the triangle after a rotation of 270° counterclockwise about the origin.

9. Find the image of $\triangle LMN$ after $Rot_{180} \circ R_{y\text{-axis}}$ if the vertices are $L(-6, 4)$, $M(-3, 2)$, and $N(-1, -2)$.

10. **Physics** The wind was blowing quite strongly when Jenny was baby-sitting. She was outside with the children, and they were throwing their large plastic ball up into the air. The wind blew the ball so that it landed approximately 3 feet east and 4 feet north of where it was thrown into the air.

   a. Make a drawing to demonstrate the original location of the ball and the translation of the ball to its landing spot.

   b. If $\begin{bmatrix} x \\ y \end{bmatrix}$ represents the original location of the ball, write a matrix that represents the location of the translated ball.

EXERCISES

Practice **Use scalar multiplication to determine the coordinates of the vertices of each dilated figure. Then graph the pre-image and the image on the same coordinate grid.**

11. triangle with vertices $A(1, 1)$, $B(1, 4)$, and $C(5, 1)$; scale factor 3

12. triangle with vertices $X(0, 8)$, $Y(-5, 9)$, and $Z(-3, 2)$; scale factor $\frac{3}{4}$

13. quadrilateral $PQRS$ with vertex matrix $\begin{bmatrix} -3 & -2 & 1 & 4 \\ 0 & 2 & 3 & 2 \end{bmatrix}$; scale factor 2

14. Graph a square with vertices $A(-1, 0)$, $B(0, 1)$, $C(1, 0)$, and $D(0, -1)$ on two separate coordinate planes.

   a. On one of the coordinate planes, graph the dilation of square $ABCD$ after a dilation of scale factor 2. Label it $A'B'C'D'$. Then graph a dilation of $A'B'C'D'$ after a scale factor of 3.

   b. On the second coordinate plane, graph the dilation of square $ABCD$ after a dilation of scale factor 3. Label it $A'B'C'D'$. Then graph a dilation of $A'B'C'D'$ after a scale factor of 2.

   c. Compare the results of parts **a** and **b.** Describe what you observe.

Use matrices to determine the coordinates of the vertices of each translated figure. Then graph the pre-image and the image on the same coordinate grid.

15. triangle *WXY* with vertex matrix $\begin{bmatrix} -2 & 1 & 3 \\ 0 & 5 & -1 \end{bmatrix}$ translated 3 units right and 2 units down

16. quadrilateral with vertices $O(0, 0)$, $P(1, 5)$, $Q(4, 7)$, and $R(3, 2)$ translated 2 units left and 1 unit down

17. square *CDEF* translated 3 units right and 4 units up if the vertices are $C(-3, 1)$, $D(1, 5)$, $E(5, 1)$, and $F(1, -3)$

18. Graph $\triangle FGH$ with vertices $F(4, 1)$, $G(0, 3)$, and $H(2, -1)$.

a. Graph the image of $\triangle FGH$ after a translation of 6 units left and 2 units down. Label the image $\triangle F'G'H'$.

b. Then translate $\triangle F'G'H'$ 1 unit right and 5 units up. Label this image $\triangle F''G''H''$.

c. What translation would move $\triangle FGH$ to $\triangle F''G''H''$ directly?

Use matrices to determine the coordinates of the vertices of each reflected figure. Then graph the pre-image and the image on the same coordinate grid.

19. $\triangle ABC$ with vertices $A(-1, -2)$, $B(0, -4)$, and $C(2, -3)$ reflected over the *x*-axis

20. $R_{y\text{-axis}}$ for a rectangle with vertices $D(2, 4)$, $E(6, 2)$, $F(3, -4)$, and $G(-1, -2)$

21. a trapezoid with vertices $H(-1, -2)$, $I(-3, 1)$, $J(-1, 5)$, and $K(2, 4)$ for a reflection over the line $y = x$

Use matrices to determine the coordinates of the vertices of each rotated figure. Then graph the pre-image and the image on the same coordinate grid.

22. Rot_{90} for $\triangle LMN$ with vertices $L(1, -1)$, $M(2, -2)$, and $N(3, -1)$

23. square with vertices $O(0, 0)$, $P(4, 0)$, $Q(4, 4)$, $R(0, 4)$ rotated 180°

24. pentagon *STUVW* with vertices $S(-1, -2)$, $T(-3, -1)$, $U(-5, -2)$, $V(-4, -4)$, and $W(-2, -4)$ rotated 270° counterclockwise

25. Proof Suppose $\triangle ABC$ has vertices $A(1, 3)$, $B(-2, -1)$, and $C(-1, -3)$. Use each result of the given transformation of $\triangle ABC$ to show how the matrix for that reflection or rotation is derived.

a. $\begin{bmatrix} 1 & -2 & -1 \\ -3 & 1 & 3 \end{bmatrix}$ under $R_{x\text{-axis}}$

b. $\begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & -3 \end{bmatrix}$ under $R_{y\text{-axis}}$

c. $\begin{bmatrix} 3 & -1 & -3 \\ 1 & -2 & -1 \end{bmatrix}$ under $R_{y=x}$

d. $\begin{bmatrix} -3 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix}$ under Rot_{90}

e. $\begin{bmatrix} -1 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix}$ under Rot_{180}

f. $\begin{bmatrix} 3 & -1 & -3 \\ -1 & 2 & 1 \end{bmatrix}$ under Rot_{270}

Given $\triangle JKL$ with vertices $J(-6, 4)$, $K(-3, 2)$, and $L(-1, -2)$. Find the coordinates of each composite transformation. Then graph the pre-image and the image on the same coordinate grid.

26. rotation of 180° followed by a translation 2 units left 5 units up

27. $R_{y\text{-axis}} \circ R_{x\text{-axis}}$

28. $Rot_{90} \circ R_{y\text{-axis}}$

Applications and Problem Solving

Real World Application

29. **Games** Each of the pieces on the chess board has a specific number of spaces and direction it can move. Research the game of chess and describe the possible movements for each piece as a translation matrix.

    a. bishop   b. knight   c. king

30. **Critical Thinking** Show that a dilation with scale factor of -1 is the same result as Rot_{180}.

31. **Entertainment** The Ferris Wheel first appeared at the 1893 Chicago Exposition. Its axle was 45 feet long. Spokes radiated from it that supported 36 wooden cars, which could hold 60 people each. The diameter of the wheel itself was 250 feet. Suppose the axle was located at the origin. Find the coordinates of the car located at the loading platform. Then find the location of the car at the 90° counterclockwise, 180°, and 270° counterclockwise rotation positions.

32. **Critical Thinking** $R_{y\text{-axis}}$ gives a matrix for reflecting a figure over the y-axis. Do you think a matrix that would represent a reflection over the line $y = 4$ exists? If so, make a conjecture and verify it.

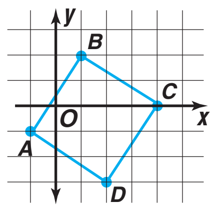
33. **Animation** Divide two sheets of grid paper into fourths by halving the length and width of the paper. Draw a simple figure on one of the pieces. On another piece, draw the figure dilated with a scale factor of 1.25. On a third piece, draw the original figure dilated with a scale factor of 1.5. On the fourth piece, draw the original figure dilated with a scale factor of 1.75. Continue dilating the original figure on each of the remaining pieces by an increase of 0.25 in scale factor each time. Put the pieces of paper in order and flip through them. What type of motion does the result of these repeated dilations animate?

34. **Critical Thinking** Write the vertex matrix for the figure graphed below.

    a. Make a conjecture about the resulting figure if you multiply the vertex matrix by $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

    b. Copy the figure on grid paper and graph the resulting vertex matrix after the multiplication described.

    c. How does the result compare with your conjecture? This is often called a *shear*. Why do you think it has this name?



Mixed Review

35. Find $A + B$ if $A = \begin{bmatrix} 3 & 8 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ -2 & 8 \end{bmatrix}$. *(Lesson 2-3)*

36. Solve the system of equations. *(Lesson 2-2)*

$x - 2y = 4.6$

$y - z = -5.6$

$x + y + z = 1.8$

37. Sales The Grandview Library holds an annual book sale to raise funds and dispose of excess books. One customer bought 4 hardback books and 7 paperbacks for $5.75. The next customer paid $4.25 for 3 hardbacks and 5 paperbacks. What are the prices for hardbacks and for paperbacks? *(Lesson 2-1)*

38. Of (0, 0), (3, 2), (−4, 2), or (−2, 4), which satisfy $x + y \geq 3$? *(Lesson 1-8)*

39. Write the standard form of the equation of the line that is parallel to the graph of $y = 4x - 8$ and passes through $(-2, 1)$. *(Lesson 1-5)*

40. Write the slope-intercept form of the equation of the line that passes through the point at (1, 6) and has a slope of 2. *(Lesson 1-3)*

41. If $f(x) = x^3$ and $g(x) = x^2 - 3x + 7$, find $(f \cdot g)(x)$ and $\left(\frac{f}{g}\right)(x)$. *(Lesson 1-2)*

42. SAT/ACT Practice If $2x + y = 12$ and $x + 2y = -6$, find the value of $2x + 2y$.

A 0 **B** 4 **C** 8 **D** 12 **E** 14

MID-CHAPTER QUIZ

1. Use graphing to solve the system of equations $\frac{1}{2}x + 5y = 17$ and $3x + 2y = 18$. (Lesson 2-1)

2. Solve the system of equations $4x + y = 8$ and $6x - 2y = -9$ algebraically. (Lesson 2-1)

3. Sales HomeMade Toys manufactures solid pine trucks and cars and usually sells four times as many trucks as cars. The net profit from each truck is $6 and from each car is $5. If the company wants a total profit of $29,000, how many trucks and cars should they sell? (Lesson 2-1)

Solve each system of equations. (Lesson 2-2)

4. $2x + y + 4z = 13$
$3x - y - 2z = -1$
$4x + 2y + z = 19$

5. $x + y = 1$
$2x - y = -2$
$4x + y + z = 8$

6. Find the values of x and y for which the matrix equation $\begin{bmatrix} y - 3 \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$ is true. (Lesson 2-3)

Use matrices *A* and *B* to find each of the following. If the matrix does not exist, write *impossible*. (Lesson 2-3)

$A = \begin{bmatrix} 3 & 5 & -7 \\ -1 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 8 & 6 \\ 5 & -9 & 10 \end{bmatrix}$

7. $A + B$ **8.** BA **9.** $B - 3A$

10. What is the result of reflecting a triangle with vertices at $A(a, d)$, $B(b, e)$, and $C(c, f)$ over the x-axis and then reflecting the image back over the x-axis? Use matrices to justify your answer. (Lesson 2-4)

Extra Practice See p. A28.

HISTORY of MATHEMATICS

MATRICES

Computers use matrices to solve many types of mathematical problems, but matrices have been around for a long time.

Early Evidence Around 300 B.C., as evidenced by clay tablets found by archaeologists, the Babylonians solved problems that now can be solved by using a system of linear equations. However, the exact method of solution used by the Babylonians has not been determined.

Margaret H. Wright

About 100 B.C.–50 B.C., in ancient China, Chapter 8 of the work *Jiuzhang suanshu (Nine Chapters of the Mathematical Art)* presented a similar problem and showed a solution on a counting board that resembles an augmented coefficient matrix.

There are three types of corn, of which three bundles of the first type, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second, and three of the third make 26 measures. How many measures of grain are contained in one bundle of each type?

Author's table

1	2	3
2	3	2
3	1	1
26	34	39

The Chinese author goes on to detail how each column can be operated on to determine the solution. This method was later credited to **Carl Friedrich Gauss.**

The Renaissance The concept of the determinant of a matrix, which you will learn about in the next lesson, appeared in Europe and Japan at almost identical times. However, **Seki** of Japan wrote about it first in 1683 with his *Method of Solving the Dissimulated Problems*. Seki's work contained matrices written in table form like those found in the ancient Chinese writings. Seki developed the pattern for determinants for 2×2, 3×3, 4×4, and 5×5 matrices and used them to solve equations, but not systems of linear equations.

In the same year in Hanover (now Germany), **Gottfried Leibniz** wrote to **Guillaume De l'Hôpital** who lived in Paris, France, about a method he had for solving a system of equations in the form $C + Ax + By = 0$. His method later became known as **Cramer's Rule.**

Modern Era In 1850, the word *matrix* was first used by **James Joseph Sylvester** to describe the tabular array of numbers. Sylvester actually was a lawyer who studied mathematics as a hobby. He shared his interests with **Arthur Cayley,** who is credited with the first published reference to the inverse of a matrix.

Today, computer experts like **Margaret H. Wright** use matrices to solve problems that involve thousands of variables. In her job as Distinguished Member of Technical Staff at a telecommunications company, she applies linear algebra for the solution of real-world problems.

ACTIVITIES

1. Solve the problem from the *Jiuzhang suanshu* by using a system of equations.
2. Research the types of problems solved by the Babylonians using a system of equations.
3. interNET CONNECTION Find out more about the personalities referenced in this article and others who contributed to the history of matrices. Visit **www.amc.glencoe.com**

CONTENTS

2-5

Determinants and Multiplicative Inverses of Matrices

OBJECTIVES
- Evaluate determinants.
- Find inverses of matrices.
- Solve systems of equations by using inverses of matrices.

Real World Application

INVESTMENTS Marshall plans to invest \$10,500 into two different bonds in order to spread out his risk. The first bond has an annual return of 10%, and the second bond has an annual return of 6%. If Marshall expects an 8.5% return from the two bonds, how much should he invest into each bond? *This problem will be solved in Example 5.*

This situation can be described by a system of equations represented by a matrix. You can solve the system by writing and solving a matrix equation.

The term determinant is often used to mean the value of the determinant.

Each square matrix has a **determinant**. The determinant of $\begin{bmatrix} 8 & 4 \\ 7 & 6 \end{bmatrix}$ is a number denoted by $\begin{vmatrix} 8 & 4 \\ 7 & 6 \end{vmatrix}$ or det $\begin{bmatrix} 8 & 4 \\ 7 & 6 \end{bmatrix}$. The value of a second-order determinant is defined as follows. *A matrix that has a nonzero determinant is called nonsingular.*

Second-Order Determinant	The value of det $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, or $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, is $a_1b_2 - a_2b_1$.

Example 1 **Find the value of $\begin{vmatrix} 8 & 4 \\ 7 & 6 \end{vmatrix}$.**

$$\begin{vmatrix} 8 & 4 \\ 7 & 6 \end{vmatrix} = 8(6) - 7(4) \text{ or } 20$$

The **minor** of an element of any nth-order determinant is a determinant of order $(n - 1)$. This minor can be found by deleting the row and column containing the element.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{The minor of } a_1 \text{ is } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

One method of evaluating an *n*th-order determinant is expanding the determinant by minors. The first step is choosing a row, any row, in the matrix. At each position in the row, multiply the *element* times its *minor* times its *position sign,* and then add the results together for the whole row. The position signs in a matrix are alternating positives and negatives, beginning with a positive in the first row, first column.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Third-Order Determinant	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Example 2 **Find the value of** $\begin{vmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{vmatrix}$.

Graphing Calculator Tip

You can use the **det(** option in the **MATH** listings of the **MATRX** menu to find a determinant.

$$\begin{vmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{vmatrix} = -4 \begin{vmatrix} -1 & 3 \\ 4 & -3 \end{vmatrix} - (-6) \begin{vmatrix} 5 & 3 \\ -2 & -3 \end{vmatrix} + 2 \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix}$$

$$= -4(-9) + 6(-9) + 2(18)$$
$$= 18$$

For any $m \times m$ matrix, the identity matrix, I, must also be an $m \times m$ matrix.

The **identity matrix for multiplication** for any square matrix A is the matrix I, such that $IA = A$ and $AI = A$. A second-order matrix can be represented by $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Since $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix for multiplication for any second-order matrix.

Identity Matrix for Multiplication	The identity matrix of nth order, I_n, is the square matrix whose elements in the main diagonal, from upper left to lower right, are 1s, while all other elements are 0s.

Multiplicative inverses exist for some matrices. Suppose A is equal to $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, a nonzero matrix of second order.

The term inverse matrix generally implies the multiplicative inverse of a matrix.

The **inverse matrix** A^{-1} can be designated as $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$. The product of a matrix A and its inverse A^{-1} must equal the identity matrix, I, for multiplication.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From the previous matrix equation, two systems of linear equations can be written as follows.

$$a_1x_1 + b_1x_2 = 1 \qquad a_1y_1 + b_1y_2 = 0$$
$$a_2x_1 + b_2x_2 = 0 \qquad a_2y_1 + b_2y_2 = 1$$

By solving each system of equations, values for x_1, x_2, y_1, and y_2 can be obtained.

$$x_1 = \frac{b_2}{a_1b_2 - a_2b_1} \qquad y_1 = \frac{-b_1}{a_1b_2 - a_2b_1}$$
$$x_2 = \frac{-a_2}{a_1b_2 - a_2b_1} \qquad y_2 = \frac{a_1}{a_1b_2 - a_2b_1}$$

If a matrix A has a determinant of 0 then A^{-1} does not exist.

The denominator $a_1b_2 - a_2b_1$ is equal to the determinant of A. If the determinant of A is not equal to 0, the inverse exists and can be defined as follows.

Inverse of a Second-Order Matrix

If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then $A^{-1} = \dfrac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$.

$A \cdot A^{-1} = A^{-1} \cdot A = I$, where I is the identity matrix.

Example 3 **Find the inverse of the matrix $\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$.**

Graphing Calculator Appendix

For keystroke instruction on how to find the inverse of a matrix, see pages A16-A17.

First, find the determinant of $\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$.

$$\begin{vmatrix} 2 & -3 \\ 4 & 4 \end{vmatrix} = 2(4) - 4(-3) \text{ or } 20$$

The inverse is $\dfrac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$. *Check to see if $A \cdot A^{-1} = A^{-1} \cdot A = I$.*

Just as you can use the multiplicative inverse of 3 to solve $3x = -27$, you can use a matrix inverse to solve a matrix equation in the form $AX = B$. To solve this equation for X, multiply each side of the equation by the inverse of A. *When you multiply each side of a matrix equation by the same number or matrix, be sure to place the number or matrix on the left or on the right on each side of the equation to maintain equality.*

$$AX = B$$
$$A^{-1}AX = A^{-1}B \quad \text{Multiply each side of the equation by } A^{-1}.$$
$$IX = A^{-1}B \quad A^{-1} \cdot A = I$$
$$X = A^{-1}B \quad IX = X$$

Example 4 **Solve the system of equations by using matrix equations.**

$$2x + 3y = -17$$
$$x - y = 4$$

Write the system as a matrix equation.

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -17 \\ 4 \end{bmatrix}$$

To solve the matrix equation, first find the inverse of the coefficient matrix.

$$\frac{1}{\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} \qquad \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 2(-1) - (1)(3) \text{ or } -5$$

Now multiply each side of the matrix equation by the inverse and solve.

$$-\frac{1}{5}\begin{bmatrix}-1 & -3\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}2 & 3\\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix} = -\frac{1}{5}\begin{bmatrix}-1 & -3\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}-17\\ 4\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-1\\ -5\end{bmatrix}$$

The solution is $(-1, -5)$.

Example 5 Real World Application

INVESTMENTS **Refer to the application at the beginning of the lesson. How should Marshall divide his \$10,500 investment between the bond with a 10% annual return and a bond with a 6% annual return so that he has a combined annual return on his investments of 8.5%?**

First, let x represent the amount to invest in the bond with an annual return of 10%, and let y represent the amount to invest in the bond with a 6% annual return. So, $x + y = 10{,}500$ since Marshall is investing \$10,500.

Write an equation in standard form that represents the amounts invested in both bonds and the combined annual return of 8.5%. That is, the amount of interest earned from the two bonds is the same as if the total were invested in a bond that earns 8.5%.

$10\%x + 6\%y = 8.5\%(x + y)$ *Interest on 10% bond = 10%x*
$0.10x + 0.06y = 0.085(x + y)$ *Interest on 6% bond = 6%y*
$0.10x + 0.06y = 0.085x + 0.085y$ *Distributive Property*
$0.015x - 0.025y = 0$
$3x - 5y = 0$ *Multiply by 200 to simplify the coefficients.*

Now solve the system of equations $x + y = 10{,}500$ and $3x - 5y = 0$. Write the system as a matrix equation and solve.

$x + y = 10{,}500$
$3x - 5y = 0$ → $\begin{bmatrix}1 & 1\\ 3 & -5\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}10{,}500\\ 0\end{bmatrix}$

Multiply each side of the equation by the inverse of the coefficient matrix.

$$-\frac{1}{8}\begin{bmatrix}-5 & -1\\ -3 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 3 & -5\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix} = -\frac{1}{8}\begin{bmatrix}-5 & -1\\ -3 & 1\end{bmatrix}\cdot\begin{bmatrix}10{,}500\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}6562.5\\ 3937.5\end{bmatrix}$$

The solution is (6562.5, 3937.5). So, Marshall should invest \$6562.50 in the bond with a 10% annual return and \$3937.50 in the bond with a 6% annual return.

CHECK FOR UNDERSTANDING

Read and study the lesson to answer each question.

1. **Describe** the types of matrices that are considered to be nonsingular.
2. **Explain** why the matrix $\begin{bmatrix} 3 & 2 & 0 \\ 4 & -3 & 5 \end{bmatrix}$ does not have a determinant. Give another example of a matrix that does not have a determinant.
3. **Describe** the identity matrix under multiplication for a fourth-order matrix.
4. **Write** an explanation as to how you can decide whether the system of equations, $ax + cy = e$ and $bx + dy = f$, has a solution.

Guided Practice

Find the value of each determinant.

5. $\begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix}$
6. $\begin{vmatrix} 12 & -26 \\ -15 & 32 \end{vmatrix}$
7. $\begin{vmatrix} 4 & 1 & 0 \\ 5 & -15 & -1 \\ -2 & 10 & 7 \end{vmatrix}$
8. $\begin{vmatrix} 6 & 4 & -1 \\ 0 & 3 & 3 \\ -9 & 0 & 0 \end{vmatrix}$

Find the inverse of each matrix, if it exists.

9. $\begin{bmatrix} -2 & 3 \\ 5 & 7 \end{bmatrix}$
10. $\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$

Solve each system of equations by using a matrix equation.

11. $5x + 4y = -3$
    $-3x - 5y = -24$
12. $6x - 3y = 63$
    $5x - 9y = 85$

13. **Metallurgy** Aluminum alloy is used in airplane construction because it is strong and lightweight. A metallurgist wants to make 20 kilograms of aluminum alloy with 70% aluminum by using two metals with 55% and 80% aluminum content. How much of each metal should she use?

EXERCISES

Practice

Find the value of each determinant.

14. $\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}$
15. $\begin{vmatrix} -4 & -1 \\ 0 & -1 \end{vmatrix}$
16. $\begin{vmatrix} 9 & 12 \\ 12 & 16 \end{vmatrix}$
17. $\begin{vmatrix} -2 & 3 \\ -2 & 1 \end{vmatrix}$
18. $\begin{vmatrix} 13 & 7 \\ -5 & -8 \end{vmatrix}$
19. $\begin{vmatrix} -6 & 5 \\ 0 & -8 \end{vmatrix}$
20. $\begin{vmatrix} 4 & -1 & -2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$
21. $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 0 & -2 \\ 1 & -3 & 0 \end{vmatrix}$
22. $\begin{vmatrix} 8 & 9 & 3 \\ 3 & 5 & 7 \\ -1 & 2 & 4 \end{vmatrix}$
23. $\begin{vmatrix} 4 & 6 & 7 \\ 3 & -2 & -4 \\ 1 & 1 & 1 \end{vmatrix}$
24. $\begin{vmatrix} 25 & 36 & 15 \\ 31 & -12 & -2 \\ 17 & 15 & 9 \end{vmatrix}$
25. $\begin{vmatrix} 1.5 & -3.6 & 2.3 \\ 4.3 & 0.5 & 2.2 \\ -1.6 & 8.2 & 6.6 \end{vmatrix}$
26. Find det A if $A = \begin{bmatrix} 0 & 1 & -4 \\ 3 & 2 & 3 \\ 8 & -3 & 4 \end{bmatrix}$.

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CONTENTS

Find the inverse of each matrix, if it exists.

27. $\begin{bmatrix} 2 & -3 \\ -2 & -2 \end{bmatrix}$

28. $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$

29. $\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$

30. $\begin{bmatrix} 6 & 7 \\ -6 & 7 \end{bmatrix}$

31. $\begin{bmatrix} -4 & 6 \\ 8 & -12 \end{bmatrix}$

32. $\begin{bmatrix} 9 & 13 \\ 27 & 36 \end{bmatrix}$

33. What is the inverse of $\begin{bmatrix} \frac{3}{4} & -\frac{1}{8} \\ 5 & \frac{1}{2} \end{bmatrix}$?

Solve each system by using a matrix equation.

34. $4x - y = 1$
$x + 2y = 7$

35. $9x - 6y = 12$
$4x + 6y = -12$

36. $x + 5y = 26$
$3x - 2y = -41$

37. $4x + 8y = 7$
$3x - 3y = 0$

38. $3x - 5y = -24$
$5x + 4y = -3$

39. $9x + 3y = 1$
$5x + y = 1$

Solve each matrix equation. The inverse of the coefficient matrix is given.

40. $\begin{bmatrix} 3 & -2 & 3 \\ 1 & 2 & 2 \\ -2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$, if the inverse is $\frac{1}{9}\begin{bmatrix} -4 & 1 & -10 \\ -3 & 3 & -3 \\ 5 & 1 & 8 \end{bmatrix}$.

41. $\begin{bmatrix} -6 & 5 & 3 \\ 9 & -2 & -1 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ -1 \end{bmatrix}$, if the inverse is $-\frac{1}{9}\begin{bmatrix} -1 & -2 & 1 \\ -12 & -15 & 21 \\ 15 & 21 & -33 \end{bmatrix}$.

Graphing Calculator

Use a graphing calculator to find the value of each determinant.

42. $\begin{vmatrix} -2 & -4 & 2 & -3 \\ 2 & 3 & 6 & 0 \\ 0 & 9 & 4 & -5 \\ 4 & -7 & 1 & 8 \end{vmatrix}$

43. $\begin{vmatrix} 2 & -9 & 1 & 8 & 4 \\ -10 & -1 & 2 & 7 & 0 \\ 0 & 4 & -6 & 1 & -8 \\ 6 & -14 & 11 & 0 & 3 \\ 5 & 1 & -3 & 2 & -1 \end{vmatrix}$

Use the algebraic methods you learned in this lesson and a graphing calculator to solve each system of equations.

44. $0.3x + 0.5y = 4.74$
$12x - 6.5y = -1.2$

45. $x - 2y + z = 7$
$6x + 2y - 2z = 4$
$4x + 6y + 4z = 14$

Applications and Problem Solving

Real World Application

46. **Industry** The Flat Rock auto assembly plant in Detroit, Michigan, produces three different makes of automobiles. In 1994 and 1995, the plant constructed a total of 390,000 cars. If 90,000 more cars were made in 1994 than in 1995, how many cars were made in each year?

47. **Critical Thinking** Demonstrate that the expression for A^{-1} is the multiplicative inverse of A for any nonsingular second-order matrix.

48. **Chemistry** How many gallons of 10% alcohol solution and 25% alcohol solution should be combined to make 12 gallons of a 15% alcohol solution?

49. Critical Thinking If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, does $(A^2)^{-1} = (A^{-1})^2$? Explain.

50. Geometry The area of a triangle with vertices at (a, b), (c, d), and (e, f) can be determined using the equation $A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$. What is the area of a triangle with vertices at $(1, -3)$, $(0, 4)$, and $(3, 0)$? (*Hint:* You may need to use the absolute value of the determinant to avoid a negative area.)

51. Retail Suppose that on the first day of a sale, a store sold 38 complete computer systems and 53 printers. During the second day, 22 complete systems and 44 printers were sold. On day three of the sale, the store sold 21 systems and 26 printers. Total sales for these items for the three days were \$49,109, \$31,614, and \$26,353 respectively. What was the unit cost of each of these two selected items?

52. Education The following type of problem often appears on placement tests or college entrance exams.

Jessi has a total of 179 points on her last two history tests. The second test score is a 7-point improvement from the first score. What are her scores for the two tests?

Mixed Review

53. Geometry The vertices of a square are $H(8, 5)$, $I(4, 1)$, $J(0, 5)$, and $K(4, 9)$. Use matrices to determine the coordinates of the square translated 3 units left and 4 units up. *(Lesson 2-4)*

54. Multiply $\begin{bmatrix} 8 & -7 \\ -4 & 0 \end{bmatrix}$ by $\frac{3}{4}$. *(Lesson 2-3)*

55. Solve the system $x - 3y + 2z = 6$, $4x + y - z = 8$, and $-7x - 5y + 4z = -10$. *(Lesson 2-2)*

56. Graph $g(x) = -2[\![x + 5]\!]$. *(Lesson 1-7)*

57. Write the standard form of the equation of the line that is perpendicular to $y = -2x + 5$ and passes through the point at $(2, 5)$. *(Lesson 1-5)*

58. Write the point-slope form of the equation of the line that passes through the points at $(1, 5)$ and $(2, 3)$. Then write the equation in slope-intercept form. *(Lesson 1-4)*

59. Safety In 1990, the Americans with Disabilities Act (ADA) went into effect. This act made provisions for public places to be accessible to all individuals, regardless of their physical challenges. One of the provisions of the ADA is that ramps should not be steeper than a rise of 1 foot for every 12 feet of horizontal distance. *(Lesson 1-3)*

a. What is the slope of such a ramp?

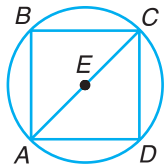
b. What would be the maximum height of a ramp 18 feet long?

60. Find $[f \circ g](x)$ and $[g \circ f](x)$ if $f(x) = x^2 + 3x + 2$ and $g(x) = x - 1$. *(Lesson 1-2)*

Extra Practice See p. A29.

61. Determine if the set of points whose coordinates are (2, 3), (−3, 4), (6, 3), (2, 4), and (−3, 3) represent a function. Explain. *(Lesson 1-1)*

62. SAT Practice The radius of circle E is 3. Square $ABCD$ is inscribed in circle E. What is the best approximation for the difference between the circumference of circle E and the perimeter of square $ABCD$?



A 3
B 2
C 1
D 0.5
E 0

CAREER CHOICES

Agricultural Manager

When you hear the word *agriculture,* you may think of a quaint little farmhouse with chickens and cows running around like in the storybooks of your childhood, but today Old McDonald's farm is big business. Agricultural managers guide and assist farmers and ranchers in maximizing their profits by overseeing the day-to-day activities. Their duties are as varied as there are types of farms and ranches.

An agricultural manager may oversee one aspect of the farm, as in feeding livestock on a large dairy farm, or tackle all of the activities on a smaller farm. They also may hire and supervise workers and oversee the purchase and maintenance of farm equipment essential to the farm's operation.

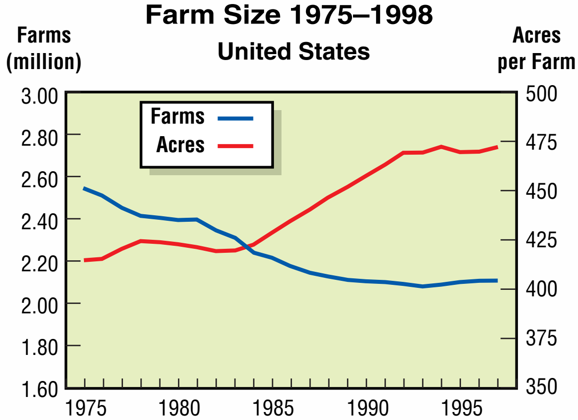
Career Overview

Degree Preferred:
Bachelor's degree in agriculture

Related Courses:
mathematics, science, finance

Outlook:
number of jobs expected to decline through 2006

Number of Farms and Average Farm Size 1975–1998



Source: NASS, Livestock & Economics Branch

*inter*NET CONNECTION For more information on careers in agriculture, visit: **www.amc.glencoe.com**

GRAPHING CALCULATOR EXPLORATION

2-5B Augmented Matrices and Reduced Row-Echelon Form

An Extension of Lesson 2-5

OBJECTIVE

- Find reduced row-echelon form of an augmented matrix to solve systems of equations.

Another way to use matrices to solve a system of equations is to use an *augmented matrix.* An augmented matrix is composed of columns representing the coefficients of each variable and the constant term.

Identify the coefficients and constants.

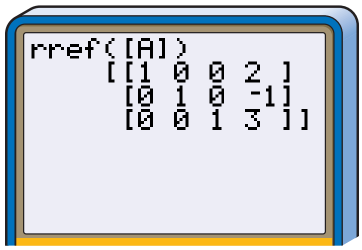
system of equations		augmented matrix
$x - 2y + z = 7$	$\mathbf{1}x - \mathbf{2}y + \mathbf{1}z = \mathbf{7}$	
$3x + y - z = 2$	$\mathbf{3}x + \mathbf{1}y - \mathbf{1}z = \mathbf{2}$	$\begin{bmatrix} 1 & -2 & 1 & 7 \\ 3 & 1 & -1 & 2 \\ 2 & 3 & 2 & 7 \end{bmatrix}$
$2x + 3y + 2z = 7$	$\mathbf{2}x + \mathbf{3}y + \mathbf{2}z = \mathbf{7}$	

Each equation is always written with the constant term on the right.

A line is often drawn to separate the constants column.

Through a series of calculations that simulate the elimination methods you used in algebraically solving a system in multiple unknowns, you can find the *reduced row-echelon form* of the matrix, which is $\begin{bmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{bmatrix}$, where c_1, c_2, and c_3 represent constants. The graphing calculator has a function **rref(** that will calculate this form once you have entered the augmented matrix. It is located in the **MATH** submenu when **MATRX** menu is accessed.

For example, if the augmented matrix above is stored as matrix A, you would enter the matrix name after the parenthesis and then insert a closing parenthesis before pressing ENTER. The result is shown at the right.



Use the following exercises to discover how this matrix is related to the solution of the system.

TRY THESE

Write an augmented matrix for each system of equations. Then find the reduced row-echelon form.

1. $2x + y - 2z = 7$
$x - 2y - 5z = -1$
$4x + y + z = -1$

2. $x + y + z - 6 = 0$
$2x - 3y + 4z - 3 = 0$
$4x - 8y + 4z - 12 = 0$

3. $w + x + y + z = 0$
$2w + x - y - z = 1$
$-w - x + y + z = 0$
$2x + y = 0$

WHAT DO YOU THINK?

4. Write the equations represented by each reduced row-echelon form of the matrix in Exercises 1-3. How do these equations related to the original system?

5. What would you expect to see on the graphing calculator screen if the constants were irrational or repeating decimals?

2-6

Solving Systems of Linear Inequalities

OBJECTIVES

- Graph systems of inequalities.
- Find the maximum or minimum value of a function defined for a polygonal convex set.

SHIPPING Package delivery services add extra charges for oversized parcels or those requiring special handling. An oversize package is one in which the sum of the length and the *girth* exceeds 84 inches. The *girth* of a package is the distance around the package. For a rectangular package, its girth is the sum of twice the width and twice the height. A package requiring special handling is one in which the length is greater than 60 inches. What size packages qualify for both oversize and special handling charges?

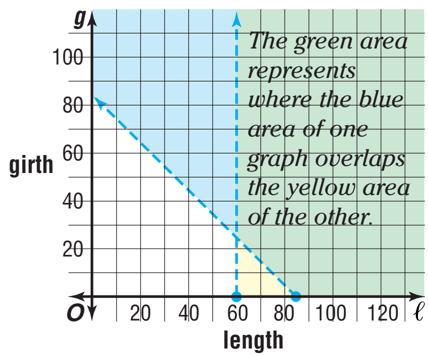
The situation described in the problem above can be modeled by a **system of linear inequalities.** To solve a system of linear inequalities, you must find the ordered pairs that satisfy both inequalities. One way to do this is to graph both inequalities on the same coordinate plane. The intersection of the two graphs contains points with ordered pairs in the solution set. If the graphs of the inequalities do not intersect, then the system has no solution.

Example 1 **SHIPPING** **What size packages qualify for both oversize and special handling charges when shipping?**

First write two inequalities that represent each type of charge. Let ℓ represent the length of a package and g represent its girth.

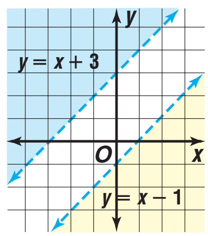
Oversize: $\ell + g > 84$

Special handling: $\ell > 60$



Neither of these inequalities includes the boundary line, so the lines are dashed. The graph of $\ell + g > 84$ is composed of all points above the line $\ell + g = 84$. The graph of $\ell > 60$ includes all points to the right of the line $\ell = 60$. The green area is the solution to the system of inequalities. That is, the ordered pair for any point in the green area satisfies both inequalities. For example, (90, 20) is a length greater than 90 inches and a girth of 20 inches which represents an oversize package that requires special handling.

Not every system of inequalities has a solution. For example, $y > x + 3$ and $y < x - 1$ are graphed at the right. Since the graphs have no points in common, there is no solution.



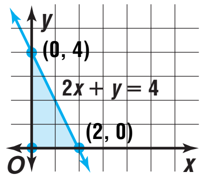
A system of more than two linear inequalities can have a solution that is a bounded set of points. A bounded set of all points on or inside a convex polygon graphed on a coordinate plane is called a **polygonal convex set.**

Example 2 **a. Solve the system of inequalities by graphing.**

$x \geq 0$
$y \geq 0$
$2x + y \leq 4$

b. Name the coordinates of the vertices of the polygonal convex set.

a. Since each inequality includes an equality, the boundary lines will be solid. The shaded region shows points that satisfy all three inequalities.



b. The region is a triangle whose vertices are the points at (0, 0), (0, 4) and (2, 0).

An expression whose value depends on two variables is a function of two variables. For example, the value of $6x + 7y - 9$ is a function of x and y and can be written $f(x, y) = 6x + 7y - 9$. The expression $f(3, 5)$ would then stand for the value of the function f when x is 3 and y is 5.

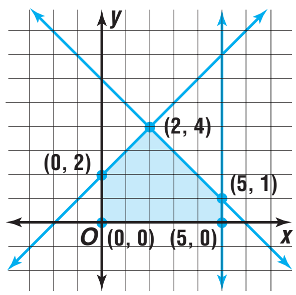
$$f(3, 5) = 6(3) + 7(5) - 9 \text{ or } 44.$$

Sometimes it is necessary to find the maximum or minimum value that a function has for the points in a polygonal convex set. Consider the function $f(x, y) = 5x - 3y$, with the following inequalities forming a polygonal convex set.

$y \geq 0$ $\quad -x + y \leq 2$ $\quad 0 \leq x \leq 5$ $\quad x + y \leq 6$

You may need to use algebraic methods to determine the coordinates of the vertices of the convex set.

By graphing the inequalities and finding the intersection of the graphs, you can determine a polygonal convex set of points for which the function can be evaluated. The region shown at the right is the polygonal convex set determined by the inequalities listed above. Since the polygonal convex set has infinitely many points, it would be impossible to evaluate the function for all of them. However, according to the **Vertex Theorem,** a function such as $f(x, y) = 5x - 3y$ need only be evaluated for the coordinates of the vertices of the polygonal convex boundary in order to find the maximum and minimum values.



Vertex Theorem

The maximum or minimum value of $f(x, y) = ax + by + c$ on a polygonal convex set occurs at a vertex of the polygonal boundary.

The value of $f(x, y) = 5x - 3y$ at each vertex can be found as follows.

$f(x, y) = 5x - 3y$
$f(0, 0) = 5(0) - 3(0) = 0$
$f(0, 2) = 5(0) - 3(2) = -6$
$f(2, 4) = 5(2) - 3(4) = -2$
$f(5, 1) = 5(5) - 3(1) = 22$
$f(5, 0) = 5(5) - 3(0) = 25$

Therefore, the maximum value of $f(x, y)$ in the polygon is 25, and the minimum is -6. The maximum occurs at (5, 0), and the minimum occurs at (0, 2).

Example 3 **Find the maximum and minimum values of $f(x, y) = x - y + 2$ for the polygonal convex set determined by the system of inequalities.**

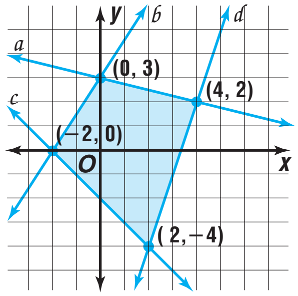
$$\mathbf{x + 4y \leq 12 \qquad 3x - 2y \geq -6 \qquad x + y \geq -2 \qquad 3x - y \leq 10}$$

First write each inequality in slope-intercept form for ease in graphing the boundaries.

*Boundary **a***	*Boundary **b***	*Boundary **c***	*Boundary **d***
$x + 4y \leq 2$	$3x - 2y \geq -6$	$x + y \geq -2$	$3x - y \leq 10$
$4y \leq -x + 12$	$-2y \geq -3x - 6$	$y \geq -x - 2$	$-y \leq -3x + 10$
$y \leq -\frac{1}{4}x + 3$	$y \leq \frac{3}{2}x + 3$		$y \geq 3x - 10$

You can use the matrix approach from Lesson 2-5 to find the coordinates of the vertices.

Graph the inequalities and find the coordinates of the vertices of the resulting polygon.



The coordinates of the vertices are $(-2, 0)$, $(2, -4)$, $(4, 2)$, $(0, 3)$.

Now evaluate the function $f(x, y) = x - y + 2$ at each vertex.

$f(-2, 0) = -2 - 0 + 2$ or 0
$f(2, -4) = 2 - (-4) + 2$ or 8
$f(4, 2) = 4 - 2 + 2$ or 4
$f(0, 3) = 0 - 3 + 2$ or -1

The maximum value of the function is 8, and the minimum value is -1.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. Refer to the application at the beginning of the lesson.
   a. **Define** the *girth* of a rectangular package.
   b. **Name** some objects that might be shipped by a package delivery service and classified as oversized *and* requiring special handling.
2. **You Decide** Marcel says there is only one vertex that will yield a maximum for any given function. Tomas says that if the numbers are correct, there could be two vertices that yield the same maximum. Who is correct? Explain your answer.
3. **Determine** how many vertices of a polygonal convex set you might expect if the system defining the set contained five inequalities, no two of which are parallel.

Guided Practice

4. Solve the system of inequalities by graphing. $x + 2y \geq 4 \qquad x - y \leq 3$
5. Solve the system of inequalities by graphing. Name the coordinates of the vertices of the polygonal convex set.
   $y \geq 0 \qquad -1 \leq x \leq 7 \qquad -x + y \leq 4 \qquad x + 2y \leq 8$

Find the maximum and minimum values of each function for the polygonal convex set determined by the given system of inequalities.

6. $f(x, y) = 4x + 3y$
   $4y \leq x + 8$
   $x + y \geq 2$
   $y \geq 2x - 5$
7. $f(x, y) = 3x - 4y$
   $x - 2y \geq -7$
   $x + y \geq 8$
   $2x - y \leq 7$

8. **Business** Gina Chuez has considered starting her own custom greeting card business. With an initial start-up cost of \$1500, she figures it will cost \$0.45 to produce each card. In order to remain competitive with the larger greeting card companies, Gina must sell her cards for no more than \$1.70 each. To make a profit, her income must exceed her costs. How many cards must she sell before making a profit?

EXERCISES

Practice

Solve each system of inequalities by graphing.

9. $y + x \geq 1$
   $y - x \leq -1$

10. $y > 1$
    $y < -3x + 3$
    $y > -3x + 1$

11. $2x + 5y < 25$
    $y < 3x - 2$
    $5x - 7y < 14$

12. Determine if $(3, -2)$ belongs to the solution set of the system of inequalities $y < \frac{1}{3}x + 5$ and $y < 2x + 1$. Verify your answer.

Solve each system of inequalities by graphing. Name the coordinates of the vertices of the polygonal convex set.

13. $y \geq -0.5x + 1$
    $y \leq -3x + 5$
    $y \leq 2x + 2$

14. $x \leq 0$
    $y + 3 \geq 0$
    $x \geq y$

15. $y \geq 0$
    $y - 5 \leq 0$
    $y + x \leq 7$
    $5x + 3y \geq 20$

16. Find the maximum and minimum values of $f(x, y) = 8x + y$ for the polygonal convex set having vertices at (0, 0), (4, 0), (3, 5), and (0, 5).

Find the maximum and minimum values of each function for the polygonal convex set determined by the given system of inequalities.

17. $f(x, y) = 3x + y$
    $x \leq 5$
    $y \geq 2$
    $2x - 5y \geq -10$

18. $f(x, y) = y - x$
    $y \leq 4 - 2x$
    $x + 2 \geq 2$
    $y \geq 0$

19. $f(x, y) = x + y$
    $y \leq 6$
    $4x - 5y \geq -10$
    $2x - 5y \leq -10$

20. $f(x, y) = 4x + 2y + 7$
    $x \geq 0$
    $y \geq 1$
    $x + y \leq 4$

21. $f(x, y) = 2x - y$
    $y \leq 4x + 6$
    $x + 4y \leq 7$
    $2x + y \leq 7$
    $x - 6y \leq 10$

22. $f(x, y) = -2x + y + 5$
    $2 \leq y \leq 8$
    $x \geq 1$
    $2x + y + 2 \leq 16$
    $y \geq 5 - x$

Applications and Problem Solving

23. **Geometry** Find the system of inequalities that will define a polygonal convex set that includes all points in the interior of a square whose vertices are $A(4, 4)$, $B(4, -4)$, $C(-4, -4)$, and $D(-4, 4)$.

24. **Critical Thinking** Write a system of more than two linear inequalities whose set of solutions is not bounded.

25. **Critical Thinking** A polygonal convex set is defined by the following system of inequalities.

    $y \leq 16 - x$ $\quad$ $3y \geq -2x + 11$ $\quad$ $y \geq 2x - 13$
    $0 \leq 2y \leq 17$ $\quad$ $y \leq 3x + 1$ $\quad$ $y \geq 7 - 2x$

    a. Determine which lines intersect and solve pairs of equations to determine the coordinates of each vertex.
    b. Find the maximum and minimum values for $f(x, y) = 5x + 6y$ in the set.

26. Business Christine's Butter Cookies sells large tins of butter cookies and small tins of butter cookies. The factory can prepare at most 200 tins of cookies a day. Each large tin of cookies requires 2 pounds of butter, and each small tin requires 1 pound of butter, with a maximum of 300 pounds of butter available each day. The profit from each day's cookie production can be estimated by the function $f(x, y) = \$6.00x + \$4.80y$, where x represents the number of large tins sold and y the number of small tins sold. Find the maximum profit that can be expected in a day.

27. Fund-raising The Band Boosters want to open a craft bazaar to raise money for new uniforms. Two sites are available. A Main Street site costs \$10 per square foot per month. The other site on High Street costs \$20 per square foot per month. Both sites require a minimum rental of 20 square feet. The Main Street site has a potential of 30 customers per square foot, while the High Street site could see 40 customers per square foot. The budget for rental space is \$1200 per month. The Band Boosters are studying their options for renting space at both sites.

a. Graph the polygonal convex region represented by the cost of renting space.

b. Determine what function would represent the possible number of customers per square foot at both locations.

c. If space is rented at both sites, how many square feet of space should the Band Boosters rent at each site to maximize the number of potential customers?

d. Suppose you were president of the Band Boosters. Would you rent space at both sites or select one of the sites? Explain your answer.

28. Culinary Arts A gourmet restaurant sells two types of salad dressing, garlic and raspberry, in their gift shop. Each batch of garlic dressing requires 2 quarts of oil and 2 quarts of vinegar. Each batch of raspberry dressing requires 3 quarts of oil and 1 quart of vinegar. The chef has 18 quarts of oil and 10 quarts of vinegar on hand for making the dressings that will be sold in the gift shop that week. If x represents the number of batches of garlic dressing sold and y represents the batches of raspberry dressing sold, the total profits from dressing sold can be expressed by the function $f(x, y) = 3x + 2y$.

a. What do you think the 3 and 2 in the function $f(x, y) = 3x + 2y$ represent?

b. How many batches of each types of dressing should the chef make to maximize the profit on sales of the dressing?

Mixed Review

29. Find the inverse of $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$. *(Lesson 2-5)*

30. Graph $y < -2x + 8$. *(Lesson 1-8)*

31. Scuba Diving Graph the equation $d + 33 = 33p$, which relates atmospheres of pressure p to ocean depth d in feet. *(Lesson 1-3)*

32. State the domain and range of the relation $\{(16, -4), (16, 4)\}$. Is this relation a function? Explain. *(Lesson 1-1)*

33. SAT Practice **Grid-In** What is the sum of four integers whose mean is 15?

Extra Practice See p. A29.

2-7

Linear Programming

OBJECTIVES

- Use linear programming procedures to solve applications.
- Recognize situations where exactly one solution to a linear programming application may not exist.

Real World Application

MILITARY SCIENCE When the U.S. Army needs to determine how many soldiers or officers to put in the field, they turn to mathematics. A system called the *Manpower Long-Range Planning System* (MLRPS) enables the army to meet the personnel needs for 7- to 20-year planning periods. Analysts are able to effectively use the MLRPS to simulate gains, losses, promotions, and reclassifications. This type of planning requires solving up to 9,060 inequalities with 28,730 variables! However, with a computer, a problem like this can be solved in less than five minutes.

The Army's MLRPS uses a procedure called **linear programming.** Many practical applications can be solved by using this method. The nature of these problems is that certain **constraints** exist or are placed upon the variables, and some function of these variables must be maximized or minimized. The constraints are often written as a system of linear inequalities.

The following procedure can be used to solve linear programming applications.

Linear Programming Procedure

1. Define variables.
2. Write the constraints as a system of inequalities.
3. Graph the system and find the coordinates of the vertices of the polygon formed.
4. Write an expression whose value is to be maximized or minimized.
5. Substitute values from the coordinates of the vertices into the expression.
6. Select the greatest or least result.

In Lesson 2-6, you found the maximum and minimum values for a given function in a defined polygonal convex region. In linear programming, you must use your reasoning abilities to determine the function to be maximized or minimized and the constraints that form the region.

Example Real World Application

1 **MANUFACTURING** **Suppose a lumber mill can turn out 600 units of product each week. To meet the needs of its regular customers, the mill must produce 150 units of lumber and 225 units of plywood. If the profit for each unit of lumber is $30 and the profit for each unit of plywood is $45, how many units of each type of wood product should the mill produce to maximize profit?**

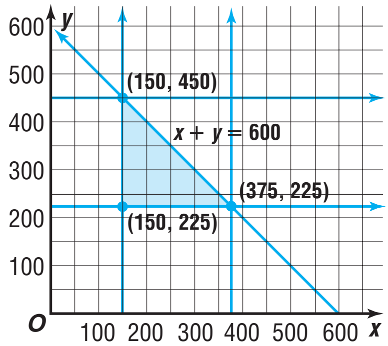
Define variables. Let x = the units of lumber produced.
Let y = the units of plywood produced.

Write inequalities. $x \geq 150$ *There cannot be less than 150 units of lumber produced.*

$y \geq 225$ *There cannot be less than 225 units of plywood produced.*

$x + y \leq 600$ *The maximum number of units produced is 600.*

Graph the system.



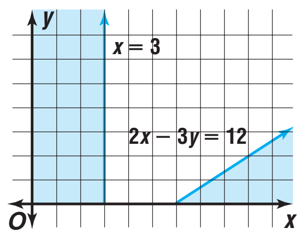
The vertices are at (150, 225), (375, 225), and (150, 450).

Write an expression. Since profit is \$30 per unit of lumber and \$45 per unit of plywood, the profit function is $P(x, y) = 30x + 45y$.

Substitute values.
$P(150, 225) = 30(150) + 45(225)$ or 14,625
$P(375, 225) = 30(375) + 45(225)$ or 21,375
$P(150, 450) = 30(150) + 45(450)$ or 24,750

Answer the problem. The maximum profit occurs when 150 units of lumber are produced and 450 units of plywood are produced.

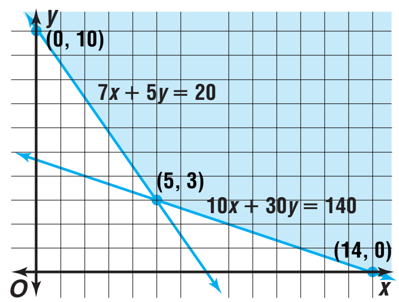
In certain circumstances, the use of linear programming is not helpful because a polygonal convex set is not defined. Consider the graph at the right, based on the following constraints.



$x \geq 0$
$y \geq 0$
$x \leq 3$
$2x - 3y \geq 12$

The constraints do not define a region with any points in common in Quadrant I. When the constraints of a linear programming application cannot be satisfied simultaneously, then the problem is said to be **infeasible.**

Sometimes the region formed by the inequalities in a linear programming application is **unbounded.** In that case, an optimal solution for the problem may not exist. Consider the graph at the right. A function like $f(x, y) = x + 2y$ has a *minimum* value at (5, 3), but it is not possible to find a maximum value.



It is also possible for a linear programming application to have two or more optimal solutions. When this occurs, the problem is said to have **alternate optimal solutions.** This usually occurs when the graph of the function to be maximized or minimized is parallel to one side of the polygonal convex set.

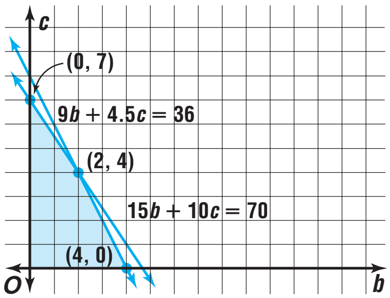
Example 2 Real World Application

SMALL BUSINESS **The Woodell Carpentry Shop makes bookcases and cabinets. Each bookcase requires 15 hours of woodworking and 9 hours of finishing. The cabinets require 10 hours of woodworking and 4.5 hours of finishing. The profit is \$60 on each bookcase and \$40 on each cabinet. There are 70 hours available each week for woodworking and 36 hours available for finishing. How many of each item should be produced in order to maximize profit?**

Define variables. Let b = the number of bookcases produced.
Let c = the number of cabinets produced.

Write inequalities.
$b \geq 0, c \geq 0$ *There cannot be less than 0 bookcases or cabinets.*
$15b + 10c \leq 70$ *No more than 70 hours of woodworking are available.*
$9b + 4.5c \leq 36$ *No more than 36 hours of finishing are available.*

Graph the system.



The vertices are at (0, 7), (2, 4), (4, 0), and (0, 0).

Write an expression. Since profit on each bookcase is \$60 and the profit on each cabinet is \$40, the profit function is $P(b, c) = 60b + 40c$.

Substitute values.
$P(0, 0) = 60(0) + 40(0)$ or 0
$P(0, 7) = 60(0) + 40(7)$ or 280
$P(2, 4) = 60(2) + 40(4)$ or 280
$P(4, 0) = 60(4) + 40(0)$ or 240

Answer the problem. The problem has alternate optimal solutions. The shop will make the same profit if they produce 2 bookcases and 4 cabinets as it will from producing 7 cabinets and no bookcases.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why the inequalities $x \geq 0$ and $y \geq 0$ are usually included as constraints in linear programming applications.
2. **Discuss** the difference between the graph of the constraints when a problem is infeasible and a graph whose constraints yield an unbounded region.
3. **Write,** in your own words, the steps of the linear programming procedure.

Guided Practice

4. Graph the system of inequalities. In a problem asking you to find the maximum value of $f(x, y)$, state whether this situation is *infeasible,* has *alternate optimal solutions,* or is *unbounded.* Assume that $x \geq 0$ and $y \geq 0$.
   $0.5x + 1.5y \geq 7$
   $3x + 9y \leq 2$
   $f(x, y) = 30x + 20y$
5. **Transportation** A package delivery service has a truck that can hold 4200 pounds of cargo and has a capacity of 480 cubic feet. The service handles two types of packages: small, which weigh up to 25 pounds each and are no more than 3 cubic feet each; and large, which are 25 to 50 pounds each and are 3 to 5 cubic feet each. The delivery service charges $5 for each small package and $8 for each large package. Let x be the number of small packages and y be the number of large packages in the truck.
   a. Write an inequality to represent the weight of the packages in pounds the truck can carry.
   b. Write an inequality to represent the volume, in cubic feet, of packages the truck can carry.
   c. Graph the system of inequalities.
   d. Write a function that represents the amount of money the delivery service will make on each truckload.
   e. Find the number of each type of package that should be placed on a truck to maximize revenue.
   f. What is the maximum revenue per truck?
   g. In this situation, is maximizing the revenue necessarily the best thing for the company to do? Explain.

Solve each problem, if possible. If not possible, state whether the problem is *infeasible*, has *alternate optimal solutions,* or is *unbounded.*

6. **Business** The manager of a gift store is printing brochures and fliers to advertise sale items. Each brochure costs 8¢ to print, and each flier costs 4¢ to print. A brochure requires 3 pages, and a flier requires 2 pages. The manager does not want to use more than 600 pages, and she needs at least 50 brochures and 150 fliers. How many of each should she print to minimize the cost?
7. **Manufacturing** Woodland Bicycles makes two models of off-road bicycles: the Explorer, which sells for $250, and the Grande Expedition, which sells for $350. Both models use the same frame, but the painting and assembly time required for the Explorer is 2 hours, while the time is 3 hours for the Grande Expedition. There are 375 frames and 450 hours of labor available for production. How many of each model should be produced to maximize revenue?

8. **Business** The Grainery Bread Company makes two types of wheat bread, light whole wheat and regular whole wheat. A loaf of light whole wheat bread requires 2 cups of flour and 1 egg. A loaf of regular whole wheat uses 3 cups of flour and 2 eggs. The bakery has 90 cups of flour and 80 eggs on hand. The profit on the light bread is \$1 per loaf and on the regular bread is \$1.50 per loaf. In order to maximize profits, how many of each loaf should the bakery make?

EXERCISES

Practice

Graph each system of inequalities. In a problem asking you to find the maximum value of $f(x, y)$, state whether the situation is *infeasible*, has *alternate optimal solutions*, or is *unbounded*. In each system, assume that $x \geq 0$ and $y \geq 0$ unless stated otherwise.

9. $y \geq 6$
$5x + 3y \leq 15$
$f(x, y) = 12x + 3y$

10. $2x + y \geq 48$
$x + 2y \geq 42$
$f(x, y) = 2x + y$

11. $4x + 3y \geq 12$
$y \leq 3$
$x \leq 4$
$f(x, y) = 3 + 3y$

Applications and Problem Solving

Real World Application

12. **Veterinary Medicine** Dr. Chen told Miranda that her new puppy needs a diet that includes at least 1.54 ounces of protein and 0.56 ounce of fat each day to grow into a healthy dog. Each cup of Good Start puppy food contains 0.84 ounce of protein and 0.21 ounce of fat. Each cup of Sirius puppy food contains 0.56 ounce of protein and 0.49 ounce of fat. If Good Start puppy food costs 36¢ per cup and Sirius costs 22¢ per cup, how much of each food should Miranda use in order to satisfy the dietary requirements at the minimum cost?
   a. Write an inequality to represent the ounces of protein required.
   b. Write an inequality to represent the ounces of fat required.
   c. Graph the system of inequalities.
   d. Write a function to represent the daily cost of puppy food.
   e. How many cups of each type of puppy food should be used in order to minimize the cost?
   f. What is the minimum cost?

13. **Management** Angela's Pizza is open from noon to midnight each day. Employees work 8-hour shifts from noon to 8 P.M. or 4 P.M. to midnight. The store manager estimates that she needs at least 5 employees from noon to 4 P.M., at least 14 employees from 4 P.M. to 8 P.M., and 6 employees from 8 P.M. to midnight. Employees are paid \$5.50 per hour for hours worked between noon and 4 P.M. The hourly pay between 4 P.M. and midnight is \$7.50.
   a. Write inequalities to represent the number of day-shift workers, the number of night-shift workers, and the total number of workers needed.
   b. Graph the system of inequalities.
   c. Write a function to represent the daily cost of payroll.
   d. Find the number of day-shift workers and night-shift workers that should be scheduled to minimize the cost.
   e. What is the minimal cost?

Solve each problem, if possible. If not possible, state whether the problem is infeasible, has alternate optimal solutions, or is unbounded.

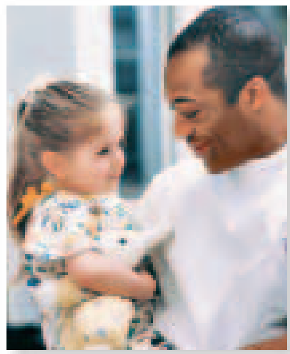
14. **Agriculture** The county officials in Chang Qing County, China used linear programming to aid the farmers in their choices of crops and other forms of agricultural production. This led to a 12% increase in crop profits, a 54% increase in animal husbandry profits, while improving the region's ecology. Suppose an American farmer has 180 acres on which to grow corn and soybeans. He is planting at least 40 acres of corn and 20 acres of soybeans. Based on his calculations, he can earn $150 per acre of corn and $250 per acre of soybeans.
    a. If the farmer plants at least 2 acres of corn for every acre of soybeans, how many acres of each should he plant to earn the greatest profit?
    b. What is the farmer's maximum profit?

15. **Education** Ms. Carlyle has written a final exam for her class that contains two different sections. Questions in section I are worth 10 points each, and questions in section II are worth 15 points each. Her students will have 90 minutes to complete the exam. From past experience, she knows that on average questions from section I take 6 minutes to complete and questions from section II take 15 minutes. Ms. Carlyle requires her students to answer at least 2 questions from section II. Assuming they answer correctly, how many questions from each section will her students need to answer to get the highest possible score?

16. **Manufacturing** Newline Recyclers processes used aluminum into food or drink containers. The recycling plant processes up to 1200 tons of aluminum per week. At least 300 tons must be processed for food containers, while at least 450 tons must be processed for drink containers. The profit is $17.50 per ton for processing food containers and $20 per ton for processing drink containers. What is the profit if the plant maximizes processing?

17. **Investment** Diego wants to invest up to $11,000 in certificates of deposit at First Bank and City Bank. He does not want to deposit more than $7,500 at First Bank. He will deposit at least $1,000 but not more than $7,000 at City Bank. First Bank offers 6% simple interest on deposits, while City Bank offers $6\frac{1}{2}$% simple interest. How much should Diego deposit into each account so he can earn the most interest possible in one year?

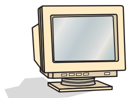
18. **Human Resources** Memorial Hospital wants to hire nurses and nurse's aides to meet patient needs at minimum cost. The average annual salary is $35,000 for a nurse and $18,000 for a nurse's aide. The hospital can hire up to 50 people, but needs to hire at least 20 to function properly. The head nurse wants at least 12 aides, but the number of nurses must be at least twice the number of aides to meet state regulations. How many nurses and nurse's aides should be hired to minimize salary costs?



19. **Manufacturing** A potato chip company makes chips to fill snack-size bags and family-size bags. In one week, production cannot exceed 2,400 units, of which at least 600 units must be for snack-size bags and at least 900 units must be for family size. The profit on a unit of snack-size bags is $12, and the profit on a unit of family-size bags is $18. How much of each type of bag must be processed to maximize profits?

20. Crafts Shelly is making soap and shampoo for gifts. She has 48 ounces of lye and 76 ounces of coconut oil and an ample supply of the other needed ingredients. She plans to make as many batches of soap and shampoo as possible. A batch of soap requires 12 ounces of lye and 20 ounces of coconut oil. Each batch of shampoo needs 6 ounces of lye and 8 ounces of coconut oil. What is the maximum number of batches of *both* soap and shampoo possible?

21. Manufacturing An electronics plant makes standard and large computer monitors on three different machines. The profit is \$40 on each monitor. Use the table below to determine how many of each monitor the plant should make to maximize profits.



Manufacturing Analysis

Machine	Hours Needed to Make: Small Monitor	Hours Needed to Make: Large Monitor	Total Hours Available
A	1	2	16
B	1	1	9
C	1	4	24

22. Critical Thinking Find the area enclosed by the polygonal convex set defined by the system of inequalities $y \geq 0$, $x \leq 12$, $2x + 6y \leq 84$, $2x - 3y \leq -3$, and $8x + 3y \geq 33$.

23. Critical Thinking Mr. Perez has an auto repair shop. He offers two bargain maintenance services, an oil change and a tune-up. His profit is \$12 on an oil change and \$20 on a tune-up. It takes Mr. Perez 30 minutes to do an oil change and 1 hour to do a tune-up. He wants to do at least 25 oil changes per week and no more than 10 tune-ups per week. He can spend up to 30 hours each week on these two services.

a. What is the most Mr. Perez can earn performing these services?

b. After seeing the results of his linear program, Mr. Perez decides that he must make a larger profit to keep his business growing. How could the constraints be modified to produce a larger profit?

Mixed Review

24. A polygonal convex set is defined by the inequalities $y \leq -x + 5$, $y \leq x + 5$, and $y + 5x \geq -5$. Find the minimum and maximum values for $f(x, y) = \frac{1}{3}x - \frac{1}{2}y$ within the set. *(Lesson 2-6)*

25. Find the values of x and y for which $\begin{bmatrix} 4x + y \\ x \end{bmatrix} = \begin{bmatrix} 6 \\ 2y - 12 \end{bmatrix}$ is true. *(Lesson 2-3)*

26. Graph $y = 3|x - 2|$. *(Lesson 1-7)*

27. Budgeting Martina knows that her monthly telephone charge for local calls is \$13.65 (excluding tax) for service allowing 30 local calls plus \$0.15 for each call after 30. Write an equation to calculate Martina's monthly telephone charges and find the cost if she made 42 calls. *(Lesson 1-4)*

28. SAT/ACT Practice If $\frac{2x - 3}{x} = \frac{3 - x}{2}$, which could be a value for x?

A -3 **B** -1 **C** 37 **D** 5 **E** 15

Extra Practice See p. A29.

VOCABULARY

additive identity matrix (p. 80)
column matrix (p. 78)
consistent (p. 67)
dependent (p. 67)
determinant (p. 98)
dilation (p. 88)
dimensions (p. 78)
element (p. 78)
elimination method (p. 68)
equal matrices (p. 79)
identity matrix for multiplication (p. 99)
image (p. 88)
inconsistent (p. 67)
independent (p. 67)
infeasible (p. 113)
inverse matrix (p. 99)
$m \times n$ matrix (p. 78)
matrix (p. 78)
minor (p. 98)
*n*th order (p. 78)
ordered triple (p. 74)
polygonal convex set (p. 108)
pre-image (p. 88)
reflection (p. 88)
reflection matrix (p. 89)
rotation (p. 88)
rotation matrix (p. 91)
row matrix (p. 78)
scalar (p. 80)
solution (p. 67)
square matrix (p. 78)
substitution method (p. 68)
system of equations (p. 67)
system of linear inequalities (p. 107)
transformation (p. 88)
translation (p. 88)
translation matrix (p. 88)
vertex matrix (p. 88)
Vertex Theorem (p. 108)
zero matrix (p. 80)

Modeling

alternate optimal solutions (p. 114)
constraints (p. 112)
linear programming (p. 112)
unbounded (p. 113)

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term from the list to complete each sentence.

1. Sliding a polygon from one location to another without changing its size, shape, or orientation is called a(n) __?__ of the figure.
2. Two matrices can be __?__ if they have the same dimensions.
3. The __?__ of $\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}$ is -5.
4. A(n) __?__ system of equations has no solution.
5. The process of multiplying a matrix by a constant is called __?__.
6. The matrices $\begin{bmatrix} 2x \\ 0 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} 8-y \\ y \\ 4x \end{bmatrix}$ are __?__ if $x = 4$ and $y = 0$.
7. A region bounded on all sides by intersecting linear inequalities is called a(n) __?__.
8. A rotation of a figure can be achieved by consecutive __?__ of the figure over given lines.
9. The symbol a_{ij} represents a(n) __?__ of a matrix.
10. Two matrices can be __?__ if the number of columns in the first matrix is the same as the number of rows in the second matrix.

added
consistent
determinant
dilation
divided
element
equal matrices
identity matrices
inconsistent
inverse
multiplied
polygonal convex set
reflections
scalar multiplication
translation

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES	REVIEW EXERCISES

Lesson 2-1 Solve systems of two linear equations.

Solve the system of equations.

$y = -x + 2$

$3x + 4y = 2$

Substitute $-x + 2$ for y in the second equation.

$$\begin{aligned} 3x + 4y &= 2 \\ 3x + 4(-x + 2) &= 2 \\ x &= 6 \end{aligned} \qquad \begin{aligned} y &= -x + 2 \\ y &= -6 + 2 \\ y &= -4 \end{aligned}$$

The solution is $(6, -4)$.

Solve each system of equations algebraically.

11. $2y = -4x$, $y = -x - 2$

12. $y = x - 5$, $6y - x = 0$

13. $2x = 5y$, $3y + x = -1$

14. $y = 6x + 1$, $2y - 15x = -4$

15. $3x - 2y = -1$, $2x + 5y = 12$

16. $x + 5y = 20.5$, $3y - x = 13.5$

Lesson 2-2 Solve systems of equations involving three variables algebraically.

Solve the system of equations.

$x - y + z = 5$

$-3x + 2y - z = -8$

$2x + y + z = 4$

Combine pairs of equations to eliminate z.

$$\begin{aligned} x - y + z &= 5 \\ -3x + 2y - z &= -8 \\ \hline -2x + y &= -3 \end{aligned} \qquad \begin{aligned} -3x + 2y - z &= -8 \\ 2x + y + z &= 4 \\ \hline -x + 3y &= -4 \end{aligned}$$

$-5y = 5$

$y = -1$

$-2x + (-1) = -3 \rightarrow x = 1$

$(1) - (-1) + z = 5 \rightarrow z = 3$ *Solve for z.*

The solution is $(1, -1, 3)$.

Solve each system of equations algebraically.

17. $x - 2y - 3z = 2$, $-3x + 5y + 4z = 0$, $x - 4y + 3z = 14$

18. $-x + 2y - 6z = 4$, $x + y + 2z = 3$, $2x + 3y - 4z = 5$

19. $x - 2y + z = 7$, $3x + y - z = 2$, $2x + 3y + 2z = 7$

Lesson 2-3 Add, subtract, and multiply matrices.

Find the difference.

$$\begin{bmatrix} -4 & 7 \\ 6 & -3 \end{bmatrix} - \begin{bmatrix} 8 & -3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 - 8 & 7 - (-3) \\ 6 - 2 & -3 - 0 \end{bmatrix} = \begin{bmatrix} -12 & 10 \\ 4 & -3 \end{bmatrix}$$

Find the product.

$$-2\begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2(1) & -2(-4) \\ -2(0) & -2(3) \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 0 & -6 \end{bmatrix}$$

Use matrices A, B, and C to find each of the following. If the matrix does not exist, write *impossible*.

$$A = \begin{bmatrix} 7 & 8 \\ 0 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -3 & -5 \\ 2 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

20. $A + B$

21. $B - A$

22. $3B$

23. $-4C$

24. AB

25. CB

26. $4A - 4B$

27. $AB - 2C$

OBJECTIVES AND EXAMPLES

Lesson 2-4 Use matrices to determine the coordinates of polygons under a given transformation.

Reflections

$R_{x\text{-axis}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$R_{y\text{-axis}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Rotations (counterclockwise about the origin)

$Rot_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$Rot_{180} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$Rot_{270} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

REVIEW EXERCISES

Use matrices to perform each transformation. Then graph the pre-image and image on the same coordinate grid.

28. $A(-4, 3)$, $B(2, 1)$, $C(5, -3)$ translated 4 units down and 3 unit left

29. $W(-2, -3)$, $X(-1, 2)$, $Y(0, 4)$, $Z(1, -2)$ reflected over the x-axis

30. $D(2, 3)$, $E(2, -5)$, $F(-1, -5)$, $G(-1, 3)$ rotated 180° about the origin.

31. $P(3, -4)$, $Q(1, 2)$, $R(-1, 1)$ dilated by a scale factor of 0.5

32. triangle ABC with vertices at $A(-4, 3)$, $B(2, 1)$, $C(5, -3)$ after $Rot_{90} \circ R_{x\text{-axis}}$

33. What translation matrix would yield the same result on a triangle as a translation 6 units up and 4 units left followed by a translation 3 units down and 5 units right?

OBJECTIVES AND EXAMPLES

Lesson 2-5 Evaluate determinants.

Find the value of $\begin{vmatrix} -7 & 6 \\ 5 & -2 \end{vmatrix}$.

$$\begin{vmatrix} -7 & 6 \\ 5 & -2 \end{vmatrix} = (-7)(-2) - 5(6) = -16$$

REVIEW EXERCISES

Find the value of each determinant.

34. $\begin{vmatrix} -3 & 5 \\ -4 & 7 \end{vmatrix}$

35. $\begin{vmatrix} 8 & -4 \\ -6 & 3 \end{vmatrix}$

36. $\begin{vmatrix} 3 & -1 & 4 \\ 5 & -2 & 6 \\ 7 & 3 & -4 \end{vmatrix}$

37. $\begin{vmatrix} 5 & 0 & -4 \\ 7 & 3 & -1 \\ 2 & -2 & 6 \end{vmatrix}$

38. Determine whether $\begin{bmatrix} 2 & -4 & 1 \\ 3 & 8 & -2 \end{bmatrix}$ has a determinant. If so, find the value of the determinant. If not, explain.

OBJECTIVES AND EXAMPLES

Lesson 2-5 Find the inverse of a 2 × 2 matrix.

Find X^{-1}, if $X = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$.

$\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -6 - (-1)$ or -5 and $-5 \neq 0$

$$X^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix}$$

REVIEW EXERCISES

Find the inverse of each matrix, if it exists.

39. $\begin{bmatrix} 3 & 8 \\ -1 & 5 \end{bmatrix}$

40. $\begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$

41. $\begin{bmatrix} -3 & 5 \\ 1 & -4 \end{bmatrix}$

42. $\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}$

43. $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$

44. $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 2-5 Solve systems of equations by using inverses of matrices.

- Solve the system of equations $3x - 5y = 1$ and $-2x + 2y = -2$ by using a matrix equation.

$$\begin{bmatrix} 3 & -5 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$-\frac{1}{4}\begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

REVIEW EXERCISES

Solve each system by using a matrix equation.

45. $2x + 5y = 1$
$-x - 3y = 2$

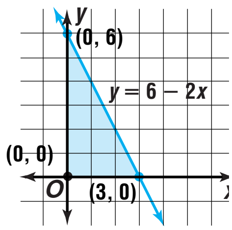
46. $3x + 2y = -3$
$-6x + 4y = 6$

47. $-3x + 5y = 1$
$-2x + 4y = -2$

48. $4.6x - 2.7y = 8.4$
$2.9x + 8.8y = 74.61$

Lesson 2-6 Find the maximum or minimum value of a function defined for a polygonal convex set.

- Find the maximum and minimum values of $f(x, y) = 4y + x - 3$ for the polygonal convex set graphed at the right.



$f(0, 0) = 4(0) + 0 - 3 = -3$ *minimum*
$f(3, 0) = 4(0) + 3 - 3 = 0$
$f(0, 6) = 4(6) + 0 - 3 = 21$ *maximum*

Find the maximum and minimum values of each function for the polygonal convex set determined by the given system of inequalities.

49. $f(x, y) = 2x + 3y$
$x \geq 1$
$y \geq -2$
$y + x \leq 6$
$y \leq 10 - 2x$

50. $f(x, y) = 3x + 2y + 1$
$x \geq 0$
$y \geq 4$
$y + x \leq 11$
$2y + x \leq 18$
$x \leq 6$

Lesson 2-7 Use linear programming procedures to solve applications.

- **Linear Programming Procedure**
  1. Define variables.
  2. Write the constraints as a system of inequalities.
  3. Graph the system and find the coordinates of the vertices of the polygon formed.
  4. Write an expression to be maximized or minimized.
  5. Substitute values from the coordinates of the vertices into the expression.
  6. Select the greatest or least result.

Use linear programming to solve.

51. Transportation Justin owns a truck and a motorcycle. He can buy up to 28 gallons of gasoline for both vehicles. His truck gets 22 miles per gallon and holds up to 25 gallons of gasoline. His motorcycle gets 42 miles per gallon and holds up to 6 gallons of gasoline. How many gallons of gasoline should Justin put in each vehicle if he wants to travel the most miles possible?

APPLICATIONS AND PROBLEM SOLVING

52. Sports In a three-team track meet, the following numbers of first-, second-, and third-place finishes were recorded.

School	First Place	Second Place	Third Place
Broadman	2	5	5
Girard	8	2	3
Niles	6	4	1

Use matrix multiplication to find the final scores for each school if 5 points are awarded for a first place, 3 for second place, and 1 for third place. (Lesson 2-4)

53. Geometry The perimeter of a triangle is 83 inches. The longest side is three times the length of the shortest side and 17 inches more than one-half the sum of the other two sides. Use a system of equations to find the length of each side. (Lesson 2-5)

54. Manufacturing A toy manufacturer produces two types of model spaceships, the *Voyager* and the *Explorer.* Each toy requires the same three operations. Each *Voyager* requires 5 minutes for molding, 3 minutes for machining, and 5 minutes for assembly. Each *Explorer* requires 6 minutes for molding, 2 minutes for machining, and 18 minutes for assembly. The manufacturer can afford a daily schedule of not more than 4 hours for molding, 2 hours for machining, and 9 hours for assembly. (Lesson 2-7)

a. If the profit is $2.40 on each *Voyager* and $5.00 on each *Explorer,* how many of each toy should be produced for maximum profit?

b. What is the maximum daily profit?

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. Suppose that a quadrilateral *ABCD* has been rotated 90° clockwise about the origin twice. The resulting vertices are $A'(2, 2)$, $B'(-1, 2)$, $C'(-2, -1)$, and $D'(3, 0)$.

   a. State the original coordinates of the vertices of quadrilateral *ABCD* and how you determined them.

   b. Make a conjecture about the effect of a double rotation of 90° on any given figure.

2. If the determinant of a coefficient matrix is 0, can you use inverse matrices to solve the system of equations? Explain your answer and illustrate it with such a system of equations.

Additional Assessment See p. A57 for Chapter 2 Practice Test.

Unit 1 interNET Project

TELECOMMUNICATION

You've Got Mail!

- Research several Internet servers and make graphs that reflect the cost of using the server over a year's time period.
- Research various e-mail servers and their costs. Write and graph equations to compare the costs.
- Determine which Internet and e-mail servers best meet your needs. Write a paragraph to explain your choice. Use your graphs to support your choice.

PORTFOLIO

Devise a real-world problem that can be solved by linear programming. Define whether you are seeking a maximum or minimum. Write the inequalities that define the polygonal convex set used to determine the solution. Explain what the solution means.

Algebra Problems

About one third of SAT math problems and many ACT math problems involve algebra. You'll need to simplify algebraic expressions, solve equations, and solve word problems.

The word problems often deal with specific problems.

- consecutive integers
- age
- motion (*distance* = *rate* × *time*)
- investments (*principal* × *rate* = *interest income*)
- work
- coins
- mixtures

THE PRINCETON REVIEW

TEST-TAKING TIP

Review the rules for simplifying algebraic fractions and expressions. Try to simplify expressions whenever possible.

ACT EXAMPLE

1. $\dfrac{(xy)^3z^0}{x^3y^4} =$

A $\dfrac{1}{y}$ **B** $\dfrac{z}{y}$ **C** z **D** xy **E** xyz

HINT Review the properties of exponents.

Solution Simplify the expression. Apply the properties of exponents.

$(xy)^3 = x^3y^3$.

Any number raised to the zero power is equal to 1. Therefore, $z^0 = 1$.

Write y^4 as y^3y^1.

Simplify.

$$\frac{(xy)^3z^0}{x^3y^4} = \frac{x^3y^3 \cdot 1}{x^3y^3y^1}$$
$$= \frac{1}{y}$$

The answer is choice **A.**

SAT EXAMPLE

2. The sum of two positive consecutive integers is x. In terms of x, what is the value of the smaller of these two integers?

A $\dfrac{x}{2} - 1$ **B** $\dfrac{x-1}{2}$ **C** $\dfrac{x}{2}$

D $\dfrac{x+1}{2}$ **E** $\dfrac{x}{2} + 1$

HINT On multiple-choice questions with variables in the answer choices, you can sometimes use a strategy called "plug-in."

Solution The "plug-in" strategy uses substitution to test the choices. Suppose the two numbers were 2 and 3. Then $x = 2 + 3$ or 5. Substitute 5 for x and see which choice yields 2, the smaller of the two numbers.

Choice A: $\dfrac{5}{2} - 1$ This is not an integer.

Choice B: $\dfrac{5-1}{2} = 2$ This is the answer, but check the rest just to be sure.

Alternate Solution You can solve this problem by writing an algebraic expression for each number. Let a be the first (smallest) positive integer. Then $(a + 1)$ is the next positive integer. Write an equation for "The sum of the two numbers is x" and solve for a.

$$a + (a + 1) = x$$
$$2a + 1 = x$$
$$2a = x - 1$$
$$a = \frac{x-1}{2}$$

The answer is choice **B.**

SAT AND ACT PRACTICE

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. If the product of $(1 + 2)$, $(2 + 3)$, and $(3 + 4)$ is equal to one half the sum of 20 and x, then $x =$

   A 10 B 85 C 105 D 190 E 1,210

2. $5\frac{1}{3} - 6\frac{1}{4} = ?$

   A $-\frac{11}{12}$

   B $-\frac{1}{2}$

   C $-\frac{2}{7}$

   D $\frac{1}{2}$

   E $\frac{9}{12}$

3. Mia has a pitcher containing x ounces of root beer. If she pours y ounces of root beer into each of z glasses, how much root beer will remain in the pitcher?

   A $\frac{x}{y} + z$

   B $xy - z$

   C $\frac{x}{yz}$

   D $x - yz$

   E $\frac{x}{y} - z$

4. Which of the following is equal to 0.064?

   A $\left(\frac{1}{80}\right)^2$ B $\left(\frac{8}{100}\right)^2$ C $\left(\frac{1}{8}\right)^2$

   D $\left(\frac{2}{5}\right)^3$ E $\left(\frac{8}{10}\right)^3$

5. A plumber charges $75 for the first thirty minutes of each house call plus $2 for each additional minute that she works. The plumber charged Mr. Adams $113 for her time. For what amount of time, in minutes, did the plumber work?

   A 38 B 44 C 49 D 59 E 64

6. If $\frac{2 + x}{5 + x} = \frac{2}{5} + \frac{2}{5}$, then $x =$

   A $\frac{2}{5}$ B 1 C 2 D 5 E 10

7. Which of the following must be true?

   I. The sum of two consecutive integers is odd.

   II. The sum of three consecutive integers is even.

   III. The sum of three consecutive integers is a multiple of 3.

   A I only

   B II only

   C I and II only

   D I and III only

   E I, II, and III

8. Jose has at least one quarter, one dime, one nickel, and one penny in his pocket. If he has twice as many pennies as nickels, twice as many nickels as dimes, and twice as many dimes as quarters, then what is the least amount of money he could have in his pocket?

   A $0.41 B $0.64 C $0.71

   D $0.73 E $2.51

9. Simplify $\dfrac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2}$.

   A $\frac{27}{8}$

   B $\frac{3}{2}$

   C $\frac{2}{3}$

   D $\frac{1}{2}$

   E $\frac{1}{3}$

10. **Grid-In** At a music store, the price of a CD is three times the price of a cassette tape. If 40 CDs were sold for a total of $480 and the combined sales of CDs and cassette tapes totaled $600, how many cassette tapes were sold?

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