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7-1

Practice

Basic Trigonometric Identities

Use the given information to determine the exact trigonometric value if $0° < \theta < 90°$.

1. If $\cos \theta = \frac{1}{4}$, find $\tan \theta$.
$\sqrt{15}$

2. If $\sin \theta = \frac{2}{3}$, find $\cos \theta$.
$\frac{\sqrt{5}}{3}$

3. If $\tan \theta = \frac{7}{2}$, find $\sin \theta$.
$\frac{7\sqrt{53}}{53}$

4. If $\tan \theta = 2$, find $\cot \theta$.
$\frac{1}{2}$

Express each value as a trigonometric function of an angle in Quandrant I.

5. $\cos 892°$
$-\cos 8°$

6. $\csc 495°$
$\csc 45°$

7. $\sin \frac{23\pi}{3}$
$-\sin \frac{\pi}{3}$

Simplify each expression.

8. $\cos x + \sin x \tan x$
$\sec x$

9. $\frac{\cot A}{\tan A}$
$\cot^2 A$

10. $\sin^2 \theta \cos^2 \theta - \cos^2 \theta$
$-\cos^4 \theta$

11. ***Kite Flying*** Brett and Tara are flying a kite. When the string is tied to the ground, the height of the kite can be determined by the formula $\frac{L}{H} = \csc \theta$, where L is the length of the string and θ is the angle between the string and the level ground. What formula could Brett and Tara use to find the height of the kite if they know the value of $\sin \theta$?
$H = L \sin \theta$

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7-1

Enrichment

The Physics of Soccer

Recall from Lesson 7-1 that the formula for the maximum height h of a projectile is $h = \frac{v_0^2 \sin^2 \theta}{2g}$, where θ is the measure of the angle of elevation in degrees, v_0 is the initial velocity in feet per second, and g is the acceleration due to gravity in feet per second squared.

Solve. Give answers to the nearest tenth.

1. A soccer player kicks a ball at an initial velocity of 60 ft/s and an angle of elevation of 40°. The acceleration due to gravity is 32 ft/s^2. Find the maximum height reached by the ball.
23.2 ft

2. With what initial velocity must you kick a ball at an angle of 35° in order for it to reach a maximum height of 20 ft?
62.4 ft/s

The distance d that a projected object travels is given by the formula $d = \frac{2v_0^2 \sin \theta \cos \theta}{g}$.

3. Find the distance traveled by the ball described in Exercise 1.
110.8 ft

In order to kick a ball the greatest possible distance at a given initial velocity, a soccer player must maximize $d = \frac{2v_0^2 \sin \theta \cos \theta}{g}$. Since 2, v_0, and g are constants, this means the player must maximize $\sin \theta \cos \theta$.

$\sin 0° \cos 0°$	$= \sin 90° \cos 90° = 0$
$\sin 10° \cos 10°$	$= \sin 80° \cos 80° = 0.1710$
$\sin 20° \cos 20°$	$= \sin 70° \cos 70° = 0.3214$

4. Use the patterns in the table to hypothesize a value of θ for which $\sin \theta \cos \theta$ will be maximal. Use a calculator to check your hypothesis. At what angle should the player kick the ball to achieve the greatest distance?
45°

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7-2 Practice

Verifying Trigonometric Identities

Verify that each equation is an identity.

1. $\frac{\csc x}{\cot x + \tan x} = \cos x$

$\frac{\csc x}{\cot x + \tan x} = \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}} \cdot \frac{\sin x \cos x}{\sin x \cos x} = \frac{\cos x}{\cos^2 x + \sin^2 x} = \frac{\cos x}{1} = \cos x$

2. $\frac{1}{\sin y - 1} - \frac{1}{\sin y + 1} = -2 \sec^2 y$

$\frac{1}{\sin y - 1} - \frac{1}{\sin y + 1} = \frac{\sin y + 1 - \sin y + 1}{\sin^2 y - 1} = \frac{2}{-\cos^2 y} = -2 \sec^2 y$

3. $\sin^3 x - \cos^3 x = (1 + \sin x \cos x)(\sin x - \cos x)$

$\sin^3 x - \cos^3 x = (\sin x - \cos x)(\sin^2 x + \sin x \cos x + \cos^2 x)$

$= (\sin x - \cos x)(1 + \sin x \cos x)$

$= (1 + \sin x \cos x)(\sin x - \cos x)$

4. $\tan \theta + \frac{\cos \theta}{1 + \sin \theta} = \sec \theta$

$\tan \theta + \frac{\cos \theta}{1 + \sin \theta} = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} = \frac{\sin \theta + \sin^2 \theta + \cos^2 \theta}{(\cos \theta)(1 + \sin \theta)}$

$= \frac{1 + \sin \theta}{(\cos \theta)(1 + \sin \theta)} = \sec \theta$

Find a numerical value of one trigonometric function of x.

5. $\sin x \cot x = 1$

$\cos x = 1$

6. $\sin x = 3 \cos x$

$\tan x = 3$

7. $\cos x = \cot x$

$\csc x = 1$ or $\sin x = 1$

8. ***Physics*** The work done in moving an object is given by the formula $W = Fd \cos \theta$, where d is the displacement, F is the force exerted, and θ is the angle between the displacement and the force. Verify that $W = Fd \frac{\cot \theta}{\csc \theta}$ is an equivalent formula.

$W = Fd \frac{\cot \theta}{\csc \theta} = Fd \frac{\frac{\cos \theta}{\sin \theta}}{\frac{1}{\sin \theta}} = Fd \frac{\cos \theta}{\sin \theta} \cdot \sin \theta = Fd \cos \theta$

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7-2 Enrichment

Building from 1 = 1

By starting with the most fundamental identity of all, 1 = 1, you can create new identities as complex as you would like them to be.

First, think of ways to write 1 using trigonometric identities. Some examples are the following.

$1 = \cos A \sec A$

$1 = \csc^2 A - \cot^2 A$

$1 = \frac{\cos (A + 360°)}{\cos (360° - A)}$

Choose two such expressions and write a new identity.

$\cos A \sec A = \csc^2 A - \cot^2 A$

Now multiply the terms of the identity by the terms of another identity of your choosing, preferably one that will allow some simplification upon multiplication.

$\cos A \sec A = \csc^2 A - \cot^2 A$

$\times \quad \frac{\sin A}{\cos A} = \tan A$

$\sin A \sec A = \tan A \csc^2 A - \cot A$

Beginning with 1 = 1, create two trigonometric identities. **Answers will vary.**

1. ____________________

2. ____________________

Verify that each of the identities you created is an identity.

3. ____________________

4. ____________________

7-3 NAME ______________ DATE ______________ PERIOD ________

Practice

Sum and Difference Identities

Use sum or difference identities to find the exact value of each trigonometric function.

1. $\cos \frac{5\pi}{12}$ **$\frac{\sqrt{6}-\sqrt{2}}{4}$**

2. $\sin(-165°)$ **$\frac{\sqrt{2}-\sqrt{6}}{4}$**

3. $\tan 345°$ **$\sqrt{3}-2$**

4. $\csc 915°$ **$-\sqrt{6}-\sqrt{2}$**

5. $\tan\left(-\frac{7\pi}{12}\right)$ **$2+\sqrt{3}$**

6. $\sec \frac{\pi}{12}$ **$\sqrt{6}-\sqrt{2}$**

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$.

7. $\cos(x+y)$ if $\sin x = \frac{5}{13}$ and $\sin y = \frac{4}{5}$
$\frac{16}{65}$

8. $\sin(x-y)$ if $\cos x = \frac{8}{17}$ and $\cos y = \frac{3}{5}$
$\frac{13}{85}$

9. $\tan(x-y)$ if $\csc x = \frac{13}{5}$ and $\cot y = \frac{4}{3}$
$-\frac{16}{63}$

Verify that each equation is an identity.

10. $\cos(180° - \theta) = -\cos\theta$
$\cos(180° - \theta)$
$= \cos 180° \cos\theta + \sin 180° \sin\theta$
$= (-1)\cos\theta + 0 \cdot \sin\theta$
$= -\cos\theta$

11. $\sin(360° + \theta) = \sin\theta$
$\sin(360° + \theta)$
$= \sin 360° \cos\theta + \cos 360° \sin\theta$
$= 0 \cdot \cos\theta + 1 \cdot \sin\theta$
$= \sin\theta$

12. ***Physics*** Sound waves can be modeled by equations of the form $y = 20\sin(3t + \theta)$. Determine what type of interference results when sound waves modeled by the equations $y = 20\sin(3t + 90°)$ and $y = 20\sin(3t + 270°)$ are combined. (*Hint:* Refer to the application in Lesson 7-3.)
The interference is destructive. The waves cancel each other completely.

7-3 NAME ______________ DATE ______________ PERIOD ________

Enrichment

Identities for the Products of Sines and Cosines

By adding the identities for the sines of the sum and difference of the measures of two angles, a new identity is obtained.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$(\textit{i})\ \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

This new identity is useful for expressing certain products as sums.

Example **Write $\sin 3\theta \cos\theta$ as a sum.**

In the right side of identity (*i*) let $\alpha = 3\theta$ and $\beta = \theta$ so that $2\sin 3\theta\cos\theta = \sin(3\theta+\theta) + \sin(3\theta-\theta)$.
Thus, $\sin 3\theta \cos\theta = \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta$.

By subtracting the identities for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$, you obtain a similar identity for expressing a product as a difference.

$$(\textit{ii})\ \sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

Example **Verify the identity $\frac{\cos 2x \sin x}{\sin 2x \cos x} = \frac{(\sin 3x - \sin x)^2}{\sin^2 3x - \sin^2 x}$.**

In the right sides of identities (*i*) and (*ii*) let $\alpha = 2x$ and $\beta = x$. Then write the following quotient.

$$\frac{2\cos 2x \sin x}{2\sin 2x \cos x} = \frac{\sin(2x+x) - \sin(2x-x)}{\sin(2x+x) + \sin(2x-x)}$$

By simplifying and multiplying by the conjugate, the identity is verified.

$$\frac{\cos 2x \sin x}{\sin 2x \cos x} = \frac{\sin 3x - \sin x}{\sin 3x + \sin x} \cdot \frac{\sin 3x - \sin x}{\sin 3x - \sin x}$$
$$= \frac{(\sin 3x - \sin x)^2}{\sin^2 3x - \sin^2 x}$$

Complete.

1. Use the identities for $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$ to find identities for expressing the products $2\cos\alpha\cos\beta$ and $2\sin\alpha\sin\beta$ as a sum or difference.
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$
$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

2. Find the value of $\sin 105° \cos 75°$ by using the identity above.
$\sin 105° \cos 75° = \frac{1}{2}(\sin 180° + \sin 30°)$
$= \frac{1}{2}(0 + \frac{1}{2}) = \frac{1}{4}$ or 0.25

7-4

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Practice

Double-Angle and Half-Angle Identities

Use a half-angle identity to find the exact value of each function.

1. $\sin 105°$ — $\frac{\sqrt{2+\sqrt{3}}}{2}$

2. $\tan \frac{\pi}{8}$ — $\sqrt{2} - 1$

3. $\cos \frac{5\pi}{8}$ — $\frac{\sqrt{2-\sqrt{2}}}{2}$

Use the given information to find sin 2θ, cos 2θ, and tan 2θ.

4. $\sin \theta = \frac{12}{13}$, $0° < \theta < 90°$ — $\frac{120}{169}, -\frac{119}{169}, -\frac{120}{119}$

5. $\tan \theta = \frac{1}{2}$, $\pi < \theta < \frac{3\pi}{2}$ — $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$

6. $\sec \theta = -\frac{5}{2}$, $\frac{\pi}{2} < \theta < \pi$ — $-\frac{4\sqrt{21}}{25}, -\frac{17}{25}, \frac{4\sqrt{21}}{17}$

7. $\sin \theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$ — $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$

Verify that each equation is an identity.

8. $1 + \sin 2x = (\sin x + \cos x)^2$

$1 + \sin 2x \stackrel{?}{=} (\sin x + \cos x)^2$

$1 + \sin 2x \stackrel{?}{=} \sin^2 x + 2 \sin x \cos x + \cos^2 x$

$1 + \sin 2x \stackrel{?}{=} 1 + 2 \sin x \cos x$

$1 + \sin 2x = 1 + \sin 2x$

9. $\cos x \sin x = \frac{\sin 2x}{2}$

$\cos x \sin x \stackrel{?}{=} \frac{\sin 2x}{2}$

$\cos x \sin x \stackrel{?}{=} \frac{2 \sin x \cos x}{2}$

$\cos x \sin x = \cos x \sin x$

10. ***Baseball*** A batter hits a ball with an initial velocity v_0 of 100 feet per second at an angle θ to the horizontal. An outfielder catches the ball 200 feet from home plate. Find θ if the range of a projectile is given by the formula $R = \frac{1}{32}v_0^2 \sin 2\theta$.
about 20°

7-4

NAME ______________ DATE ________ PERIOD ______

Enrichment

Reading Mathematics: Using Examples

Most mathematics books, including this one, use examples to illustrate the material of each lesson. Examples are chosen by the authors to show how to apply the methods of the lesson and to point out places where possible errors can arise.

1. Explain the purpose of Example 1c in Lesson 7-4.
to illustrate how to use the double-angle identity for the tangent; to show how to find tan θ from information already known

2. Explain the purpose of Example 3 in Lesson 7-4.
to illustrate how a double-angle identity can be applied to a real-world situation

3. Explain the purpose of Example 4 in Lesson 7-4.
to illustrate the verification of a trigonometric identity involving a double-angle identity

To make the best use of the examples in a lesson, try following this procedure:

a. When you come to an example, stop. Think about what you have just read. If you don't understand it, reread the previous section.

b. Read the example problem. Then instead of reading the solution, try solving the problem yourself.

c. After you have solved the problem or gone as far as you can go, study the solution given in the text. Compare your method and solution with those of the authors. If necessary, find out where you went wrong. If you don't understand the solution, reread the text or ask your teacher for help.

4. Explain the advantage of working an example yourself over simply reading the solution given in the text.
Sample answer: This method checks your understanding of the material rather than your ability to follow the authors' logic. By allowing errors to arise in your solution, it helps you find areas of misunderstanding. Then it gives you a method for correcting your errors and checking your solution.

7-5

NAME ______________ DATE ________ PERIOD ______

Practice

Solving Trigonometric Equations

Solve each equation for principal values of x. Express solutions in degrees.

1. $\cos x = 3\cos x - 2$
0°

2. $2\sin^2 x - 1 = 0$
±45°

Solve each equation for 0° ≤ x < 360°.

3. $\sec^2 x + \tan x - 1 = 0$
0°, 135°, 180°, 315°

4. $\cos 2x + 3\cos x - 1 = 0$
60°, 300°

Solve each equation for 0 ≤ x < 2π.

5. $4\sin^2 x - 4\sin x + 1 = 0$
$\frac{\pi}{6}, \frac{5\pi}{6}$

6. $\cos 2x + \sin x = 1$
$0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$

Solve each equation for all real values of x.

7. $3\cos 2x - 5\cos x = 1$
$\frac{2\pi}{3} + 2\pi k, \frac{4\pi}{3} + 2\pi k$

8. $2\sin^2 x - 5\sin x + 2 = 0$
$\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$

9. $3\sec^2 x - 4 = 0$
$\frac{\pi}{6} + \pi k, \frac{5\pi}{6} + \pi k$

10. $\tan x(\tan x - 1) = 0$
$\pi k, \frac{\pi}{4} + \pi k$

11. ***Aviation*** An airplane takes off from the ground and reaches a height of 500 feet after flying 2 miles. Given the formula $H = d\tan\theta$, where H is the height of the plane and d is the distance (along the ground) the plane has flown, find the angle of ascent θ at which the plane took off.
about 2.7°

7-5

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Enrichment

The Spectrum

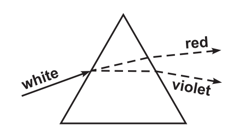
In some ways, light behaves as though it were composed of waves. The wavelength of visible light ranges from about 4×10^{-5} cm for violet light to about 7×10^{-5} cm for red light.

As light passes through a medium, its velocity depends upon the wavelength of the light. The greater the wavelength, the greater the velocity. Since white light, including sunlight, is composed of light of varying wavelengths, waves will pass through the medium at an infinite number of different speeds. The index of refraction n of the medium is defined by $n = \frac{c}{v}$, where c is the velocity of light in a vacuum (3×10^{10} cm/s), and v is the velocity of light in the medium. As you can see, the index of refraction of a medium is not a constant. It depends on the wavelength and the velocity of light passing through it. (The index of refraction of diamond given in the lesson is an average.)

1. For all media, $n > 1$. Is the speed of light in a medium greater than or less than c? Explain.
less than; $v = \frac{c}{n}$. Since $n > 1$, $v < c$.

2. A beam of violet light travels through water at a speed of 2.234×10^{10} cm/s. Find the index of refraction of water for violet light.
1.343

The diagram shows why a prism splits white light into a spectrum. Because they travel at different velocities in the prism, waves of light of different colors are refracted different amounts.



3. Beams of red and violet light strike crown glass at an angle of 20°. Use Snell's Law to find the difference between the angles of refraction of the two beams.
violet light: $n = 1.531$ red light: $n = 1.513$
about 0.16°