NAME _____ DATE _____ PERIOD _____

11-1

Practice

Rational Exponents

Evaluate each expression.

1. $\dfrac{8^{\frac{2}{3}}}{8^{\frac{1}{3}}}$ **2**

2. $\left(\dfrac{4}{5}\right)^{-2}$ $\dfrac{25}{16}$

3. $343^{\frac{2}{3}}$ **49**

4. $\sqrt[3]{8^3}$ **8**

5. $\sqrt{5} \cdot \sqrt{10}$ $5\sqrt{2}$

6. $9^{-\frac{1}{2}}$ $\dfrac{1}{3}$

Simplify each expression.

7. $(5n^3)^2 \cdot n^{-6}$ **25**

8. $\left(\dfrac{x^2}{4y^{-2}}\right)^{-\frac{1}{2}}$ $2|x|^{-1}|y|^{-1}$ or $\dfrac{2}{|x||y|}$

9. $(64x^6)^{\frac{1}{3}}$ $4x^2$

10. $(5x^6y^4)^{\frac{1}{2}}$ $|x|^3y^2\sqrt{5}$

11. $\sqrt{x^2y^3} \cdot \sqrt{x^3y^4}$ $|x|^{\frac{5}{2}}|y|^{\frac{7}{2}}$ or $x^2|y|^3\sqrt{xy}$

12. $\left(\dfrac{p^{6a}}{p^{-3a}}\right)^{\frac{1}{3}}$ p^{3a}

Express each using rational exponents.

13. $\sqrt{x^5y^6}$ $|x|^{\frac{5}{2}}|y|^3$

14. $\sqrt[5]{27x^{10}y^5}$ $27^{\frac{1}{5}}x^2y$

15. $\sqrt{144x^6y^{10}}$ $12|x|^3|y|^5$

16. $21\sqrt[3]{c^7}$ $21c^{\frac{7}{3}}$

17. $\sqrt{1024a^3}$ $32a^{\frac{3}{2}}$

18. $\sqrt[4]{36a^8b^5}$ $6^{\frac{1}{2}}a^2b^{\frac{5}{4}}$

Express each using radicals.

19. $64^{\frac{1}{3}}$ $\sqrt[3]{64}$ or **4**

20. $2^{\frac{1}{2}}a^{\frac{3}{2}}b^{\frac{5}{2}}$ $\sqrt{2a^3b^5}$ or $ab^2\sqrt{2ab}$

21. $s^{\frac{2}{3}}t^{\frac{1}{3}}v^{\frac{2}{3}}$ $\sqrt[3]{s^2tv^2}$

22. $y^{\frac{3}{2}}$ $\sqrt{y^3}$ or $y\sqrt{y}$

23. $x^{\frac{2}{5}}y^{\frac{3}{5}}$ $\sqrt[5]{x^2y^3}$

24. $(x^6y^3)^{\frac{1}{2}}z^{\frac{3}{2}}$ $\sqrt{x^6y^3z^3}$ or $|x|^3yz\sqrt{yz}$

NAME _____ DATE _____ PERIOD _____

11-1

Enrichment

Look for Cases

Many problems can be partitioned into a few cases. The cases are classifications within the problem that are exclusive one to another but which taken together comprise all the possibilities in the problem. Divisions into cases are made where the characteristics of the problem are most critical.

Example **If $a > b$ and $\dfrac{1}{a} > \dfrac{1}{b}$, what must be true of a and b?**

One of the critical characteristics of inequalities is that changing the signs of the left and right sides requires changing the sense of the inequality. This suggests classifying a and b according to signs. Since neither a nor b can equal zero in the second inequality and $a > b$, the cases are:

(i) a and b are both positive.
(ii) a and b are both negative.
(iii) a is positive and b is negative.

An important characteristic of reciprocals is that 1 and –1 are their own reciprocals. Therefore, within each of the above cases consider examples of these three subclassifications:

(A) a and b are both greater than 1 (or < –1).
(B) a and b are both fractions whose absolute values are less than one.
(C) a is greater than 1 (or –1) and b is less than 1 (or –1).

For case (i), consider these examples.

(A) $3 > 2 \rightarrow \dfrac{1}{3} < \dfrac{1}{2}$ (B) $\dfrac{1}{4} > \dfrac{1}{5} \rightarrow 4 < 5$ (C) $\dfrac{3}{2} > \dfrac{3}{4} \rightarrow \dfrac{2}{3} < \dfrac{4}{3}$

For case (ii), consider these examples.

(A) $-2 > -3 \rightarrow -\dfrac{1}{2} < -\dfrac{1}{3}$ (B) $-\dfrac{1}{5} > -\dfrac{1}{4} \rightarrow -5 < -4$

(C) $-\dfrac{3}{4} > -\dfrac{3}{2} \rightarrow -\dfrac{4}{3} < -\dfrac{2}{3}$

For case (iii), consider these examples.

(A) $2 > -3 \rightarrow \dfrac{1}{2} > -\dfrac{1}{3}$ (B) $\dfrac{1}{5} > -\dfrac{1}{4} \rightarrow 5 > -4$ (C) $\dfrac{3}{4} > -\dfrac{3}{2} \rightarrow \dfrac{4}{3} > -\dfrac{2}{3}$

Notice that the second given inequality, $\dfrac{1}{a} > \dfrac{1}{b}$, only holds true in case (iii). We conclude that a must be positive and b negative to satisfy both given inequalities.

Complete.

1. Find the positive values of b such that $b^{x_1} > b^{x_2}$ whenever $x_1 < x_2$. $0 < b < 1$

2. Solve $\dfrac{x}{x+1} > 0$. $x > 0$ or $x < -1$

3. Find the domain of $f(x) = \sqrt{3 - |x - 2|}$. $\{x \mid -1 \le x \le 5\}$

NAME _____ DATE _____ PERIOD _____

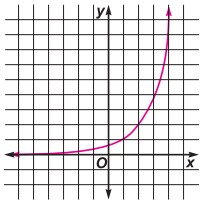
11-2

Practice

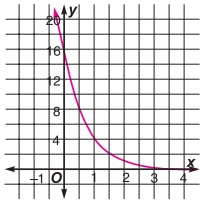
Exponential Functions

Graph each exponential function or inequality.

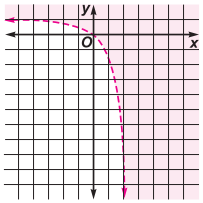
1. $y = 2^{x-1}$



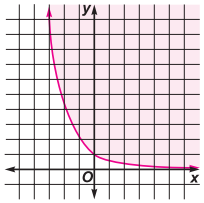
2. $y = 4^{-x+2}$



3. $y > -3^x + 1$



4. $y \geq 0.5^x$



5. ***Demographics*** An area in North Carolina known as The Triangle is principally composed of the cities of Durham, Raleigh, and Chapel Hill. The Triangle had a population of 700,000 in 1990. The average yearly rate of growth is 5.9%. Find the projected population for 2010. **about 2,203,014**

6. ***Finance*** Determine the amount of money in a savings account that provides an annual rate of 4% compounded monthly if the initial investment is $1000 and the money is left in the account for 5 years. **about $1221.00**

7. ***Investments*** How much money must be invested by Mr. Kaufman if he wants to have $20,000 in his account after 15 years? He can earn 5% compounded quarterly.
about $9491.35

NAME _____ DATE _____ PERIOD _____

11-2

Enrichment

Finding Solutions of $x^y = y^x$

Perhaps you have noticed that if x and y are interchanged in equations such as $x = y$ and $xy = 1$, the resulting equation is equivalent to the original equation. The same is true of the equation $x^y = y^x$. However, finding solutions of $x^y = y^x$ and drawing its graph is not a simple process.

Solve each problem. Assume that x and y are positive real numbers

1. If $a > 0$, will (a, a) be a solution of $x^y = y^x$? Justify your answer.
Yes, since $a^a = a^a$ must be true by the reflexive prop. of equality.

2. If $c > 0$, $d > 0$, and (c, d) is a solution of $x^y = y^x$, will (d, c) also be a solution? Justify your answer.
Yes; replacing x with d, y with c gives $d^c = c^d$; but if (c, d) is a solution, $c^d = d^c$. So, by the symmetric property of equality, $d^c = c^d$ is true.

3. Use 2 as a value for y in $x^y = y^x$. The equation becomes $x^2 = 2^x$.

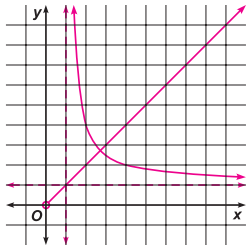
a. Find equations for two functions, $f(x)$ and $g(x)$ that you could graph to find the solutions of $x^2 = 2^x$. Then graph the functions on a separate sheet of graph paper.
$f(x) = x^2$, $g(x) = 2^x$ See students' graphs.

b. Use the graph you drew for part **a** to state two solutions for $x^2 = 2^x$. Then use these solutions to state two solutions for $x^y = y^x$.
Sample answer: 2 or 4; (2, 2), (4, 2)

4. In this exercise, a graphing calculator will be very helpful. Use the technique from Exercise 3 to complete the tables below. Then graph $x^y = y^x$ for positive values of x and y. If there are a symptotes, show them in your diagram using dotted lines. Note that in the table, some values of y call for one value of x, others call for two.

x	y
$\frac{1}{2}$	$\frac{1}{2}$
$\frac{3}{4}$	$\frac{3}{4}$
1	1
2	2
4	2
3	3
2.5	3

x	y
4	4
2	4
5	5
1.8	5
8	8
1.5	8



NAME _____ DATE _____ PERIOD _____

11-3

Practice

The Number *e*

1. ***Demographics*** In 1995, the population of Kalamazoo, Michigan, was 79,089. This figure represented a 0.4% annual decline from 1990.
   a. Let t be the number of years since 1995 and write a function that models the population in Kalamazoo in 1995.
   $y = 79{,}089e^{-0.004t}$
   b. Predict the population in 2010 and 2015. Assume a steady rate of decline. **2010: 74,483; 2015: 73,008**

2. ***Biology*** Suppose a certain type of bacteria reproduces according to the model $P(t) = 100e^{0.271t}$, where t is time in hours.
   a. At what rate does this type of bacteria reproduce?
   27.1%
   b. What was the initial number of bacteria?
   100
   c. Find the number of bacteria at $P(5)$, $P(10)$, $P(24)$, and $P(72)$. Round to the nearest whole number.
   $P(5)$: 388
   $P(10)$: 1503
   $P(24)$: 66,781
   $P(72)$: 29,782,004,910

3. ***Finance*** Suppose Karyn deposits \$1500 in a savings account that earns 6.75% interest compounded continuously. She plans to withdraw the money in 6 years to make a \$2500 down payment on a car. Will there be enough funds in Karyn's account in 6 years to meet her goal?
   No. Karyn will have \$2249 in her account in 6 years. She will be \$251 short.

4. ***Banking*** Given the original principal, the annual interest rate, the amount of time for each investment, and the type of compounded interest, find the amount at the end of the investment.
   a. $P = \$1250$, $r = 8.5\%$, $t = 3$ years, semiannually
   \$1604.60
   b. $P = \$2575$, $r = 6.25\%$, $t = 5$ years 3 months, continuously
   \$3575.03

NAME _____ DATE _____ PERIOD _____

11-3

Enrichment

Approximations for π and *e*

The following expression can be used to approximate e. If greater and greater values of n are used, the value of the expression approximates e more and more closely.

$$\left(1 + \frac{1}{n}\right)^n$$

Another way to approximate e is to use this infinite sum. The greater the value of n, the closer the approximation.

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots + \frac{1}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot n} + \ldots$$

In a similar manner, π can be approximated using an infinite product discovered by the English mathematician John Wallis (1616-1703).

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdots$$

Solve each problem.

1. Use a calculator with an e^x key to find e to 7 decimal places.
   2.7182818
2. Use the expression $\left(1 + \frac{1}{n}\right)^n$ to approximate e to 3 decimal places. Use 5, 100, 500, and 7000 as values of n.
   2.488, 2.705, 2.716, 2.718
3. Use the infinite sum to approximate e to 3 decimal places. Use the whole numbers from 3 through 6 as values of n.
   2.667, 2.708, 2.717, 2.718
4. Which approximation method approaches the value of e more quickly?
   the infinite sum
5. Use a calculator with a π key to find π to 7 decimal places.
   3.1415927
6. Use the infinite product to approximate π to 3 decimal places. Use the whole numbers from 3 through 6 as values of n.
   2.926, 2.972, 3.002, 3.023
7. Does the infinite product give good approximations for π quickly?
   no
8. Show that $\pi^4 + \pi^5$ is equal to e^6 to 4 decimal places.
   To 4 decimal places, they both equal 403.4288.
9. Which is larger, e^π or π^e?
   $e^\pi > \pi^e$
10. The expression $x^{\frac{1}{x}}$ reaches a maximum value at $x = e$. Use this fact to prove the inequality you found in Exercise 9.
   $e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$; $\left(e^{\frac{1}{e}}\right)^{\pi e} > \left(\pi^{\frac{1}{\pi}}\right)^{\pi e}$; $e^\pi > \pi^e$

NAME _____ DATE _____ PERIOD _____

11-4

Practice

Logarithmic Functions

Write each equation in exponential form.

1. $\log_3 81 = 4$
$3^4 = 81$

2. $\log_8 2 = \frac{1}{3}$
$8^{\frac{1}{3}} = 2$

3. $\log_{10} \frac{1}{100} = -2$
$10^{-2} = \frac{1}{100}$

Write each equation in logarithmic form.

4. $3^3 = 27$
$\log_3 27 = 3$

5. $5^{-3} = \frac{1}{125}$
$\log_5 \frac{1}{125} = -3$

6. $\left(\frac{1}{4}\right)^{-4} = 256$
$\log_{\frac{1}{4}} 256 = -4$

Evaluate each expression.

7. $\log_7 7^3$
3

8. $\log_{10} 0.001$
-3

9. $\log_8 4096$
4

10. $\log_4 32$
$\frac{5}{2}$

11. $\log_3 1$
0

12. $\log_6 \frac{1}{216}$
-3

Solve each equation.

13. $\log_x 64 = 3$
4

14. $\log_4 0.25 = x$
-1

15. $\log_4 (2x - 1) = \log_4 16$
$\frac{17}{2}$

16. $\log_{10} \sqrt{10} = x$
$\frac{1}{2}$

17. $\log_7 56 - \log_7 x = \log_7 4$
14

18. $\log_5 (x + 4) + \log_5 x = \log_5 12$
2

19. ***Chemistry*** How long would it take 100,000 grams of radioactive iodine, which has a half-life of 60 days, to decay to 25,000 grams? Use the formula $N = N_0\left(\frac{1}{2}\right)^t$, where N is the final amount of the substance, N_0 is the initial amount, and t represents the number of half-lives. **120 days**

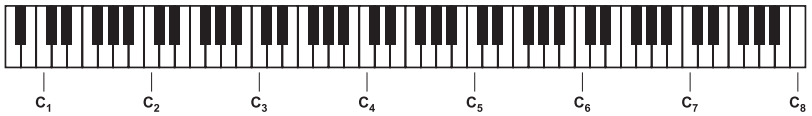
NAME _____ DATE _____ PERIOD _____

11-4

Enrichment

Musical Relationships

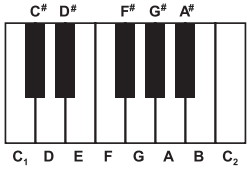
The frequencies of notes in a musical scale that are one octave apart are related by an exponential equation. For the eight C notes on a piano, the equation is $C_n = C_1 2^{n-1}$, where C_n represents the frequency of note C_n.



1. Find the relationship between C_1 and C_2. $C_2 = 2C_1$

2. Find the relationship between C_1 and C_4. $C_4 = 8C_1$

The frequencies of consecutive notes are related by a common ratio r. The general equation is $f_n = f_1 r^{n-1}$.

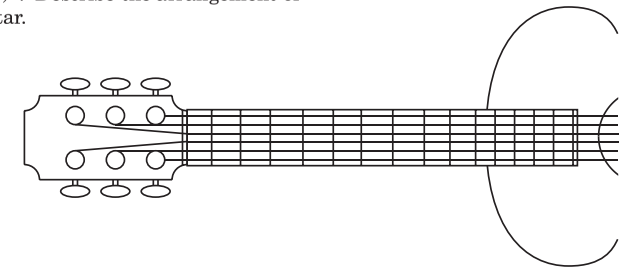


3. If the frequency of middle C is 261.6 cycles per second and the frequency of the next higher C is 523.2 cycles per second, find the common ratio r. (Hint: The two Cs are 12 notes apart.) Write the answer as a radical expression.
$r = \sqrt[12]{2}$

4. Substitute decimal values for r and f_1 to find a specific equation for f_n.
$f_n = 261.6\,(1.05946)^{n-1}$

5. Find the frequency of F# above middle C.
$f_7 = 261.6\,(1.05946)^6 \approx 369.95$

6. The frets on a guitar are spaced so that the sound made by pressing a string against one fret has about 1.0595 times the wavelength of the sound made by using the next fret. The general equation is $w_n = w_0\,(1.0595)^n$. Describe the arrangement of the frets on a guitar.
The frets are spaced in a logarithmic scale.



NAME _____ DATE _____ PERIOD _____

11-5

Practice

Common Logarithms

Given that log 3 = 0.4771, log 5 = 0.6990, and log 9 = 0.9542, evaluate each logarithm.

1. log 300,000
5.4771

2. log 0.0005
−3.3010

3. log 9000
3.9542

4. log 27
1.4313

5. log 75
1.8751

6. log 81
1.9084

Evaluate each expression.

7. log 66.3
1.8215

8. $\log \frac{17^4}{5}$
4.2228

9. $\log 7(4^3)$
2.6513

Find the value of each logarithm using the change of base formula.

10. $\log_6 832$
3.7526

11. $\log_{11} 47$
1.6056

12. $\log_3 9$
2

Solve each equation or inequality.

13. $8^x = 10$
1.1073

14. $2.4^x \leq 20$
$x \leq 3.4219$

15. $1.8^{x-5} = 19.8$
10.0795

16. $3^{5x} = 85$
0.8088

17. $4^{2x} > 25$
$x > 1.1610$

18. $3^{2x-2} = 2^x$
1.4608

19. ***Seismology*** The intensity of a shock wave from an earthquake is given by the formula $R = \log_{10} \frac{I}{I_0}$, where R is the magnitude, I is a measure of wave energy, and $I_0 = 1$. Find the intensity per unit of area for the following earthquakes.

a. Northridge, California, in 1994, $R = 6.7$
about 5,011,872

b. Hector Mine, California, in 1999, $R = 7.1$
about 12,589,254

NAME _____ DATE _____ PERIOD _____

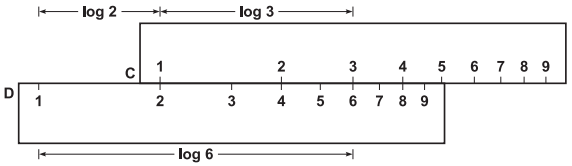
11-5

Enrichment

The Slide Rule

Before the invention of electronic calculators, computations were often performed on a slide rule. A slide rule is based on the idea of logarithms. It has two movable rods labeled with C and D scales. Each of the scales is logarithmic.

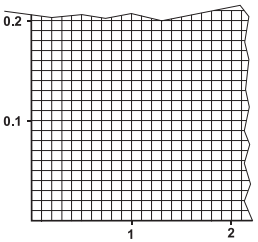
To multiply 2×3 on a slide rule, move the C rod to the right as shown below. You can find 2×3 by adding log 2 to log 3, and the slide rule adds the lengths for you. The distance you get is 0.778, or the logarithm of 6.



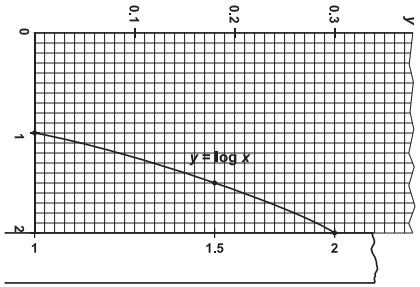
1-2 See students' work.

Follow the steps to make a slide rule.

1. Use graph paper that has small squares, such as 10 squares to the inch. Using the scales shown at the right, plot the curve $y = \log x$ for $x = 1, 1.5$, and the whole numbers from 2 through 10. Make an obvious heavy dot for each point plotted.



2. You will need two strips of cardboard. A 5-by-7 index card, cut in half the long way, will work fine. Turn the graph you made in Exercise 1 sideways and use it to mark a logarithmic scale on each of the two strips. The figure shows the mark for 2 being drawn.



3. Explain how to use a slide rule to divide 8 by 2. **Line up the 2 on the C scale with the 8 on the D scale. The quotient is the number on the D scale below the 1 on the C scale.**

NAME _____ DATE _____ PERIOD _____

11-6

Practice

Natural Logarithms

Evaluate each expression.

1. ln 71
4.2627

2. ln 8.76
2.1702

3. ln 0.532
−0.6311

4. antiln −0.256
0.7741

5. antiln 4.62
101.4940

6. antiln −1.62
0.1979

Convert each logarithm to a natural logarithm and evaluate.

7. $\log_7 94$
2.3348

8. $\log_5 256$
3.4454

9. $\log_9 0.712$
−0.1546

Use natural logarithms to solve each equation or inequality.

10. $6^x = 42$
2.0860

11. $7^x = 4^{x+3}$
7.4317

12. $1249 = 175e^{-0.04t}$
−49.1328

13. $10^{x+1} > 3^x$
$x > -1.9125$

14. $12 < e^{0.048y}$
$y > 51.7689$

15. $8.4 < e^{t-2}$
$t > 4.1282$

16. ***Banking*** Ms. Cubbatz invested a sum of money in a certificate of deposit that earns 8% interest compounded continuously. The formula for calculating interest that is compounded continuously is $A = Pe^{rt}$. If Ms. Cubbatz made the investment on January 1, 1995, and the account was worth $12,000 on January 1, 1999, what was the original amount in the account? **$8713.79**

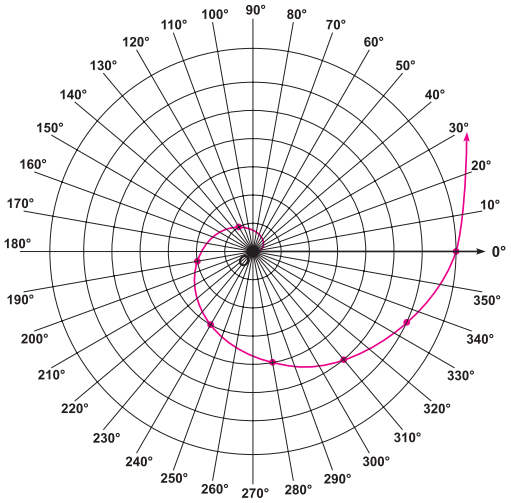
NAME _____ DATE _____ PERIOD _____

11-6

Enrichment

Spirals

Consider an angle in standard position with its vertex at a point O and its initial side on a polar axis. Remember that point P on the terminal side of the angle can be named by (r, θ), where r is the directed distance of the point from O and θ is the measure of the angle. As you learned, graphs in this system may be drawn on polar coordinate paper such as the kind shown below.



1. Use a calculator to complete the table for $\log_2 r = \frac{\theta}{120}$. (To find $\log_2 a$ on a calculator, press LOG a ÷ LOG 2). Round θ to the nearest degree if necessary.

r		1	2	3	4	5	6	7	8
θ	0°	120°	190°	240°	279°	310°	337°	360°	

2. Plot the points found in Exercise 1 on the grid above and connect them to form a smooth curve. **See graph above.**

This type of spiral is called a *logarithmic spiral* because the angle measures are proportional to the logarithms of the radii.

NAME _____ DATE _____ PERIOD _____

11-7

Practice

Modeling Real-World Data with Exponential and Logarithmic Functions

Find the amount of time required for an amount to double at the given rate if the interest is compounded continuously.

1. 4.75%
14.59 years

2. 6.25%
11.09 years

3. 5.125%
13.52 years

4. 7.1%
9.76 years

5. ***City Planning*** At a recent town council meeting, proponents of increased spending claimed that spending should be doubled because the population of the city would double over the next three years. Population statistics for the city indicate that population is growing at the rate of 16.5% per year. Is the claim that the population will double in three years correct? Explain.
No. To double in size, the population of the city would have to be increasing at the rate of 23.1% per year. The population of the city will double in 4.2 years.

6. ***Conservation*** A wildlife conservation group released 14 black bears into a protected area. Their goal is to double the population of black bears every 4 years for the next 12 years.

a. If they are to meet their goal at the end of the first four years, what should be the yearly rate of increase in population?
17.3%

b. Suppose the group meets its goal. What will be the minimum number of black bears in the protected area in 12 years?
There will be at least 112 black bears in the protected area.

c. What type of model would best represent such data?
An exponential model would best represent these data.

NAME _____ DATE _____ PERIOD _____

11-7

Enrichment

Hyperbolic Functions

The *hyperbolic functions* are a family of functions of great importance in calculus and higher-level mathematics. Because they are defined in terms of the hyperbola, their name is derived from that word. These functions have an interesting relationship to the number e and to the trigonometric functions, uniting those seemingly unrelated subjects with the conic sections.

The hyperbolic functions can be written in terms of e.

Hyperbolic sine of x: $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic tangent of x: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Identities involving hyperbolic functions exhibit strong resemblances to trigonometric identities.

Example **Show that $\sinh 2x = 2 \sinh x \cosh x$.**

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2} \quad \leftarrow \text{Replace } x \text{ in the definition above by } 2x.$$

$$= 2\left(\frac{e^{2x} - e^{-2x}}{4}\right)$$

$$= 2\left(\frac{(e^x)^2 - (e^{-x})^2}{4}\right) \quad \leftarrow \text{difference of two squares}$$

$$= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= 2 \sinh x \cosh x$$

1. Find $\cosh^2 x - \sinh^2 x$. **1**

Prove each identity.

2. $\sinh(-x) = -\sinh x$

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = \frac{-(e^x - e^{-x})}{2} = -\left(\frac{e^x - e^{-x}}{2}\right) = -\sinh x$$

3. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$$\sinh(x + y) = \frac{e^{x+y} - e^{-(x+y)}}{2}$$

$$= \frac{e^{x+y} - e^{-(x+y)}}{4} + \frac{e^{x+y} - e^{-(x+y)}}{4}$$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$= \sinh x \cosh y + \cosh x \sinh y$$